## THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and twenty-second regular meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, February 25, 1905. The attendance at the two sessions was about fifty, including the following forty-two members of the Society:

Dr. Grace Andrews, Professor C. L. Bouton, Professor Joseph Bowden, Professor E. W. Brown, Professor F. N. Cole, Professor E. S. Crawley, Miss L. D. Cummings, Dr. D. R. Curtiss, Dr. W. S. Dennett, Dr. L. P. Eisenhart, Dr. William Findlay, Professor T. S. Fiske, Dr. A. S. Gale, Miss Ida Griffiths, Professor E. R. Hedrick, Mr. E. A. Hook, Mr. S. A. Joffe, Dr. Edward Kasner, Dr. O. D. Kellogg, Professor C. J. Keyser, Mr. E. H. Koch, Professor E. O. Lovett, Dr. Emory McClintock, Dr. Max Mason, Professor James Pierpont, Miss Virginia Ragsdale, Miss Amy Rayson, Dr. F. G. Reynolds, Miss I. M. Schottenfels, Professor Charlotte A. Scott, Dr. A. W. Smith, Professor D. E. Smith, Dr. C. E. Stromquist, Professor Henry Taber, Professor J. H. Tanner, Miss M. E. Trueblood, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor L. A. Wait, Mr. H. E. Webb, Dr. E. B. Wilson, Dr. Ruth G. Wood.

The Vice-Presidents, Professors Pierpont and E. W. Brown, presided at the morning and afternoon sessions respectively. The Council announced the election of the following persons to membership in the Society: Miss A. F. Becker, Yeatman High School, St. Louis, Mo.; Professor C. H. Beckett, Purdue University; Professor W. De W. Cairns, Oberlin College; Professor S. C. Davisson, Indiana University; Dr. J. S. French, Jacob Tome Institute; Mr. F. H. Hodge, Clark University; Mr. A. E. Joslyn, Armour Institute of Technology; Dr. J. W. Lowber, Austin, Texas; Mr. J. H. Maclagan-Wedderburn, University of Chicago; Mr. G. A. Plimpton, New York City; Mr. E. W. Ponzer, University of Illinois; Mr. H. W. Reddick, University of Illinois; Miss M. E. Sinclair, University of Nebraska; Dr. A. W. Smith, Colgate University. Ten applications for membership were received.

Professor T. S. Fiske desiring to withdraw from the Editorial Committee of the *Transactions* at the completion of the present volume, the vacancy thus created was filled by the election of Professor E. B. Van Vleck.

The following papers were read at this meeting:

(1) Dr. L. D. AMES: "The theorem that a closed simple surface is bilateral."

(2) Professor C. L. BOUTON: "Note on isothermal curves and one-parameter groups of conformal transformations in the plane."

(3) Professor E. W. BROWN: "Note on the variation of the arbitrary and given constants in dynamical equations."

(4) Mr. O. E. GLENN: "Determination of the abstract groups of order $p^2qr$."

(5) Mr. F. R. SHARPE: "The stability of the motion of a viscous liquid."

(6) Professor JAMES PIERPONT: "Note on infinite products."

(7) Professor CHARLOTTE A. SCOTT: "The elementary treatment of conics by means of the regulus."

(8) Dr. A. W. SMITH: "The symbolic treatment of differential geometry."

(9) Mr. A. M. HILTEBEITEL: "Note on a problem in mechanics."

(10) Mr. R. B. ALLEN: "Hypercomplex number systems with respect to a domain of rationality."

(11) Dr. L. P. EISENHART: "Note on the deformation of surfaces of translation."

Mr. Sharpe's paper was communicated to the Society through Professor E. W. Brown, Mr. Hiltebeitel's through Professor Lovett, and Mr. Allen's through Professor Taber. Mr. Glenn was introduced by Professor Crawley. In the absence of the authors, Dr. Ames's paper was read by Professor Hedrick, Mr. Sharpe's by Professor Brown, Mr. Allen's by Professor Taber, and Mr. Hiltebeitel's paper was read by title. Professor Bouton's paper appears in the present number of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the papers in the list above.

1. Darboux (Théorie générale des surfaces, volume 1, page 361) has stated the theorem that a closed unilateral surface has a multiple point, and has suggested a proof for the case in

which the surface can be represented by an algebraic equation of the form $\phi(x, y, z) = 0$. Without assuming that it can be so represented, Dr. Ames proves in this paper that a unilateral smooth closed simple surface cannot exist. If such a surface exists, it can be cut apart and pairs of coincident conical surfaces can be joined to the free edges, so that the total surface is closed and can be treated as a bilateral surface. It is possible to choose a closed curve which cuts the original surface once and only once. By considering the order (cf. BULLETIN, November, 1904, page 581; March, 1904, page 305) of the points of this curve with respect to the enlarged surface, a contradiction is obtained which proves the theorem.

3. The *Transactions* for July, 1903, contains a paper by Professor Brown in which a certain theorem concerning the variation of the known constants in dynamical equations is enunciated. In the present note he shows that the proof of the theorem is faulty, that it is of more limited application and that it requires restrictions in addition to those set down in the previous paper. The theorem in question concerns the possibility of putting $\delta c_i = 0$ in the canonical equations of variations. However, the application to the determination of the secular acceleration of the moon's mean motion is not affected.

4. Mr. Glenn gave an enumeration of all non-isomorphic groups of order $p^2qr$, and discussed the peculiarities of a number of the types given. Among other results of the investigation the following theorem was enunciated:

A group $G_{p^2qr}$ $(p > q > r)$ always contains a self-conjugate subgroup $H$ of order $p^2q$. If $H$ is the only maximal invariant subgroup of $G$, then, except when $r = 2$, $H$ must be abelian. When $r = 2$ there exists one type if $p \equiv 1 \pmod{q}$, and one if $p \equiv -1 \pmod{q}$, whose subgroup $H_{p^2q}$ is not abelian.

An analogous theorem holds for the group $G_{p^mqr}$ whose subgroup $I_{p^m}$ is abelian of type $[1, 1, 1, \cdots (m \text{ units})]$.

5. Professor Reynolds has examined the limit of stability for the steady motion of a viscous liquid flowing symmetrically between two infinite parallel planes. Mr. Sharpe considers this problem by a simpler method and, by assuming a somewhat different type of periodic variation from the steady motion, he obtains a lower limit of stability. He also extends the method

used to a similar problem in the case of a cylindrical tube and deduces a limit of stability which is much lower than that conjectured from general considerations by Professor Reynolds.

6. The uniform and absolute convergence of the infinite product $P = f_1(z)f_2(z)\cdots$ is usually made to depend upon the series $G = g_1(z) + g_2(z) + \cdots$ where $f_n = 1 + g_n (n = 1, 2, \cdots)$. Professor Pierpont showed that this question might often be treated more simply without the intervention of the $G$ series.

7. In the usual treatment of conics in projective geometry, the point system of the second order and the line system of the second order are defined independently by means of projective flat pencils and projective ranges; the connection between the two systems is arrived at by somewhat elaborate processes. Miss Scott shows that, by means of the regulus, the point system and line system can be treated together. The polar and involution properties of conics follow immediately, and simple proofs are given for the theorems of Chasles, Brianchon, and Pascal.

8. Dr. Smith's symbolic treatment of differential geometry is based on the investigations of Professor Maschke on differential invariants* and involves notations similar in character and treatment to those employed in the theory of invariants. The quadratic forms of differential geometry are represented symbolically as perfect squares or in the case of congruences as a product of two linear factors. For the practical working of the symbols there is a list of identities more or less independent of the specific meanings of the different symbols. The principal results are these: All invariant forms are visibly such. Equations may be written in a much more compact form, for example, the differential equation of geodesics, which is $(f\phi)(\phi U)(U, (fU)) = 0$. The fundamental magnitudes, for example, the coefficients of the cubic form, are symmetrically expressible symbolically in terms of symbols of lower orders; and the proof of theorems, such as the existence theorem for a surface whose two quadratic forms are given, may be made with perfect generality as to the parameters used, and the desired derivatives actually found.

* "A new method of determining the differential parameters and invariants of quadratic differential quantics," *Trans. Am. Math. Soc.*, 1900. "Invariants of differential quantics," Decennial Publications, University of Chicago, vol. 9.

9. In an article published in the twenty-eighth volume of the *Giornale di Matematiche*, Dr. Carlo Bonacini states with inadequate proof that in the motion of a body under the influence of forces directed towards two fixed centers the variables in the equations of motion can be separated only in case the forces vary inversely as the squares of the respective distances of the body from the fixed centers. The object of Mr. Hiltebeitel's paper is to point out an error in Bonacini's reasoning and to show that the most general law of force depending only on the distance and admitting of the separation of the variables in the equations of motion is a certain linear combination of the laws of direct distance and inverse square of distance first considered by Lagrange in his memoirs on this classic particular case (first studied by Euler) of the problem of three bodies.

10. Mr. Allen's paper is concerned with two problems in the theory of hypercomplex number systems with respect to a domain of rationality:* First, the problem of the determination with respect to a given domain of rationality of all types of hypercomplex number systems in a given number of units; that is, given an arbitrary scalar domain of rationality, to determine all hypercomplex number systems, in a given number $n$ of units into one or other of which any given number system in $n$ units, whose constants of multiplication are included in $R$, can be transformed by a transformation rational with respect to $R$. This problem is solved by Mr. Allen for the domain $R \equiv R(1)$ and for $n \geqq 6$. Second, the determination of all number systems $e_1, e_2, \cdots, e_n$ whose constants of multiplication are included in an arbitrary scalar domain $R$ of rationality, and in which division is unambiguous of numbers rational with respect to the hypercomplex domain $\mathfrak{R}(R, e_i)$ of rationality, designating as rational with respect to $\mathfrak{R}(R, e_i)$ any number $\sum_1^n a_i e_i$ of the system whose coefficients $a_1, a_2$, etc., are rational with respect to $R$. Every such system contains a modulus; and the number of units is determined by the characteristic equation $A^m + p_1(a)A^{m-1} + \cdots + p_{m-1}(a)A + p_m(a) = 0$ of the system. Namely, we have $n = \mu^2\nu$ where $m = \mu\nu$, $\mu$ being any factor of $m$. We may take as units of such a system the products $A^pB^q$, $(p = 0, 1, 2, \cdots, m;\ q = 0, 1, \cdots,$

* See *Transactions*, vol. 5, p. 511.

$\mu - 1$) of powers of two properly chosen numbers $A = \sum_1^n a_i e_i$ and $B = \sum_1^n b_i e_i$ rational with respect to $\mathfrak{R}(R, e_i)$. Any such system can be transformed into one consisting of $\nu$ mutually nilfactorial quadrates of order $\mu$ by a transformation rational with respect to the domain obtained by adjoining to $R$ the roots of the equation $x^m + p_1(a)x^{m-1} + \cdots + p_{m-1}(a)x + p_m(a) = 0$ for a properly chosen number $A = \sum_1^n a_i e_i$ of the system rational with respect to $\mathfrak{R}(Re_i)$.

11. In the January number of the BULLETIN, Dr. Burke Smith proves that the minimal surfaces and surfaces of translation whose generators are in perpendicular planes are the only surfaces of translation which can be deformed in a continuous manner in such a way that the generators continue to be generators. Dr. Eisenhart applies to these surfaces a theorem due to Adam and gets pairs of applicable surfaces of translation with the generators in correspondence; and the equations of these surfaces involve seven arbitrary parameters. The conditions to be satisfied in order that the generators be plane on these new surfaces are given, and a few examples are discussed.

F. N. COLE,
*Secretary.*

---

## THE DECEMBER MEETING OF THE CHICAGO SECTION.

THE sixteenth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held in the Northwestern University Building, Chicago, on December 30 and 31, 1904. The attendance was thirty-seven, including the following members of the Society:

Mr. R. P. Baker, Professor D. P. Bartlett, Professor G. A. Bliss, Dr. W. H. Bussey, Professor Florian Cajori, Professor D. F. Campbell, Professor L. E. Dickson, Dr. E. L. Dodd, Dr. Saul Epsteen, Professor A. G. Hall, Professor Thomas F. Holgate, Mr. N. J. Lennes, Mr. E. P. Lytle, Professor Heinrich Maschke, Professor G. W. Myers, Mr. Oscar Schmiedel, Miss I. M. Schottenfels, Professor H. E. Slaught, Dr. Burke Smith, Professor E. J. Townsend, Dr. Oswald Veblen, Pro-

fessor A. L. P. Wernicke, Professor H. S. White, Mr. A. H. Wilson, Mr. N. R. Wilson, Dr. A. E. Young.

The first session opened at ten o'clock, Professor E. J. Townsend, of the University of Illinois, in the chair. The following officers were elected for the ensuing year: Secretary, Professor Thomas F. Holgate; additional members of the programme committee, Professor G. A. Bliss, Professor A. G. Hall.

The following papers were read:

(1) Professor L. E. DICKSON: "A definition of a group by independent postulates."

(2) Professor L. E. DICKSON: "Determination of the ternary modular groups."

(3) Mr. A. H. WILSON: "On the linear automorphic transformation of the binary quartic."

(4) Mr. N. J. LENNES: "Areas and volumes."

(5) Dr. SAUL EPSTEEN and Mr. J. MACLAGAN-WEDDERBURN: "On the composition of a hypercomplex number system."

(6) Professor L. E. DICKSON: "On the minimum degree of resolvents for the $p$-section of the periods of hyperelliptic functions of four periods."

(7) Professor L. E. DICKSON: "On semigroups and the general isomorphism between infinite groups."

(8) Dr. BURKE SMITH: "Deformation of surfaces with preservation of a conjugate system."

(9) Dr. OSWALD VEBLEN: "Non-metrical definition of the linear continuum."

(10) Dr. A. E. YOUNG: "Surfaces defined by the quadratic forms $\lambda(du^2 + dv^2)$ and $D(du^2 \pm dv^2) + 2D'dudv$.

(11) Professor H. S. WHITE: "Polar tetrads to a quartic surface upon a twisted cubic, with generalizations."

(12) Mr. N. J. LENNES: "On the necessary conditions for the convergence of an improper definite integral."

Some mathematical models were exhibited by Mr. R. P. Baker. Friday evening at six thirty o'clock dinner was served to those in attendance at the University Club of Chicago.

Abstracts of the papers presented follow below. The abstracts are numbered to correspond to the titles in the list above.

1. An abstract group may be defined by the four postulates: (1) product is in set; (2) composition is associative; (3) presence

of a right-hand identity $i$; (4) for a particular $i$ (if existent) there occurs a right-hand inverse of each element $a$, namely a solution of $a \circ a' = i$. The postulates are proved to be sufficient, consistent and independent. In fact, if we add a postulate fixing the order of the set or the commutative postulate, the five postulates are independent. This definition was obtained by Professor Dickson as a simplification of Professor Moore's first definition (*Transactions*, 1902). In Professor Moore's recent proof that postulate $(3''_r)$ of his second definition was redundant, he obtained relations sufficient to establish the present definition, but did not set up the definition itself, but rather a very desirable simplification of his old second definition. This paper will appear, together with Professor Moore's paper, in the *Transactions* for April, 1905.

2. The determination of all $m$-ary linear homogeneous groups of determinant unity modulo $p$ falls naturally into two problems, according as the order is a multiple of $p$ or prime to $p$. The case $m = 3$ has been discussed at length by Professor Burnside, *Proceedings of the London Mathematical Society*, volume 27 (1894), pages 58–106, with limitations as to the number of factors of $p^2 + p + 1$. An error led him to overlook the groups with an invariant ternary quadratic form. The paper by Professor Dickson gives a brief but complete determination of the ternary groups of orders multiples of $p$, by methods quite different from those employed by Professor Burnside. The paper will appear in the *American Journal of Mathematics*.

3. The transformations of the tetrahedron group are given in the literature in the canonical form. Mr. Wilson derives expressions for these transformations without specialization of the coördinate system for the quartic, making use of its irrational covariants. Incidentally the group for the harmonic case of the quartic is also given.

4. The paper points out a certain minor change necessary in Hilbert's theory of areas: In regard to volumes, it is shown that while the present theory, which is built without the use of any continuity axioms, is a complete theory in terms of measure of volume, there is no such theory in terms of geometric congruence. Thus if of two polyhedrons $P_1$ and $P_2$, $P_1$ is con-

gruent to part of $P_2$ it is known that $M(P_1)$ (where $M(P_1)$ denotes the measure of volume of $P_1$) is less than $M(P_2)$. On the other hand, if $M(P_1) < M(P_2)$ we do not know that $P_1$ is congruent to part of $P_2$.

By means of the archimedean axiom the following theorem is proved: "If $M(P_1) < M(P_2)$, then $P_1$ is congruent to part of $P_2$." It is shown that this theorem cannot be proved without the archimedean axiom. The following definitions are then given: $P_1 < P_2$ if $P_1$ is congruent to part of $P_2$. $P_1$ is equal to $P_2$ if (1) $P_1$ is not congruent to part of $P_2$ and (2) $P_2$ is not congruent to part of $P_1$. It follows that for any two polyhedrons $P_1$ and $P_2$ one of the following relations must hold: $P_1 < P_2$, $P_1 = P_2$, $P_1 > P_2$ these relations being defined in terms of geometric congruences.

5. By the product $EE_1$ of an algebra $E \equiv e_1 \cdots e_m\, e_{m+1} \cdots e_n$ by one of its subcomplexes $E_1 = e_{m+1} \cdots e_n$ is meant the totality of independent units arising when every unit of $E$ is multiplied in turn by every unit of $E_1$. If $EE_1 \preceq E_1$, $E_1E \preceq E_1$, the algebra $E_1$ is said to be invariant in $E$. According to Molien (*Mathematische Annalen*, volume 41, page 93), if $E_1$ is invariant in $E$, then there exists an algebra $K_1$ which accompanies $E$. $K_1$ is also said to be complementary to $E_1$ with respect to $E$.

The series $E, E_1, E_2, \cdots$ is said to be normal when $E_s$ is a maximal invariant subalgebra of $E_{s-1}$ ($E_0 = E$), and $K_1, K_2, \cdots$ is said to be a normal complementary series when $K_s$ is complementary to $E_s$, with respect to $E_{s-1}$. If $n_s$ is the order (number of units) of $E_s$, then the integers $l_s$ are defined as follows: $l_s = (n_s - \text{order of maximal subalgebra of } E_{s-1} \text{ containing } E_s)$, $s = 1, 2, \cdots$. In addition, the integers $k_s$ are defined thus: $k_s = n_{s-1} - n_s$.

If $C_s$ is the maximal subalgebra of $C_{s-1}$ which is invariant in $E$, then $E, C_1, C_2, \cdots$ is said to be a *chief* series of $E$. The following theorems are demonstrated.

1) If $E$ has two invariant subalgebras $E_1, E_1'$ then ($a$) $E$ is reducible in the sense that the product of any number of $E$ by any number of $E'$ is zero or belongs to their common subalgebra (Peirce, *American Journal of Mathematics*, volume 4, page 100); or else ($b$) $E_1' \preceq E_1$.

2) If $E_1$ and $E_1'$ are two maximal invariant subalgebras of $E$ and if their common part is $E_1''$, then $E_1''$ is a maximal invariant subalgebra of $E_1$ and $E_1'$.

3) The normal complementary series $K_1, K_2, \cdots$ is independent, apart from the sequence, of the normal series $E, E_1', E_2', \cdots$.

4) The sets of integers $l_1, l_2, \cdots$ and also the set of integers $k_1, k_2, \cdots$ are independent, apart from the sequence, of the normal series selected.

5) The chief complementary series $k_1^{(c)}, k_2^{(c)}, \cdots$ is independent, apart from the sequence, of the chief series $E, C_1, C_2, \cdots$.

6. The third paper by Professor Dickson is the outcome of an investigation conducted under the auspices of the Carnegie Institution of Washington. The chief result is the theorem that $\tau = (p^4 - 1)/(p - 1)$, when $p > 3$. For the case $p = 3$, the problem is equivalent to the determination of the 27 lines on a general cubic surface. Considerable headway has also been made towards the determination of all the subgroups of the quaternary linear abelian group modulo $p$, this group having the same relation to the hyperelliptic modular theory that the linear fractional congruence group bears to the elliptic modular theory. The paper was published in the *Transactions* for January, 1905.

7. The fourth paper by Professor Dickson treats of the general mutual correspondence between the elements of two groups $A$ and $B$, such that to each element of either correspond one or more elements of the other, and such that products correspond. Let $A'[B']$ be the set of elements of $A[B]$ which correspond to the identity of $B[A]$. The sets $A'$ and $B'$ are semigroups, not necessarily groups if $A$ and $B$ are infinite. This was shown by explicit examples, in one of which neither $A'$ nor $B'$ contains the identity. But if one of the semigroups $A', B'$ is a group, the other is also. If we omit his postulate that the set is finite, we obtain from Weber's definition of a group a set of independent postulates for a semigroup. A semigroup contains at most one identity and at most one inverse of any element. The paper will appear in the April number of the *Transactions*.

8. In Dr. Smith's paper the surfaces whose total curvature, in terms of the parameters of the asymptotic lines, is of the form $k = -1/[\phi(u) + \psi(v)]^2$ are classified, and their associate surfaces studied. The discussion leads to the determination of the

cartesian coördinates of a new class of surfaces depending on three arbitrary functions, which may be deformed with preservation of a conjugate system, one family of the invariant conjugate system being lines of zero length. They are the associates of a class of surfaces first considered by Stäckel, on which the two families of lines of curvature fall together into a single family.

9. Dr. Veblen's paper contains a set of independent postulates for the linear continuum. They are stated in terms of order relations alone, with no notion of laws of operation or equality of segments. The main element of novelty is a "postulate of uniformity" which replaces Cantor's assumption of a numerable everywhere dense subset and leads to the following proposition:

About every element $x$ of the continuum there is a sequence of segments $\{\sigma_k^{(x)}\}$, $(k = 1, 2, 3, \cdots)$ such that: (1) $\sigma_k^{(x)}$ includes $\sigma_{k+1}^{(x)}$; (2) $x$ is the only point that lies on every $\sigma_k^{(x)}(k = 1, 2, 3, \cdots)$; (3) if $\sigma$ is any segment whatever, there exists an $n$ (independent of $x$) such that no segment $\sigma_n^{(x)}$ contains every point of $\sigma$.

10. Dr. A. E. Young concludes the discussion of surfaces defined by the quadratic forms $\lambda(du^2+dv^2)$ and $D(du^2 \pm dv^2) + 2D'dudv$. In a paper given last spring before the Society he showed the existence of surfaces having the linear element $(u+v)^2\left(\frac{du^2}{f(u)}+\frac{dv^2}{\phi(v)}\right)$ when referred to lines of curvature. Proceeding with the discussion of the surfaces having this linear element, he finds that the cartesian coördinates may be obtained without the solution of Riccati equations as is customary. Making use of this fact, he finds the cartesian coördinates of the surfaces in question. In the most general case, at least one set of the lines of curvature are plane curves, either transcendental curves resembling somewhat the cycloids, or else the well-known nodal cubic curves. However, for certain forms of the functions involved, both sets of the lines of curvature are plane curves. In particular, there are surfaces on which both sets of the lines of curvature are nodal cubics. A model of these latter surfaces was shown.

11. A polar tetrad to a quadric surface is constituted by 4 points, every pair of which are conjugate with respect to that

surface. Can such a tetrad vary while its points remain upon a twisted cubic curve? Professor White showed that while this is not possible for every cubic and quadric, it can occur for special quadrics invariantively conditioned by the cubic curves. Given the curve and two polar tetrads upon it, the equation of the surface was explicitly determined, a surface triply tangent to the curve. The problem was extended to polar pentads of a cubic surface, to forms of all orders above the first in 4-space, and to cubics, quartics, etc., in the plane, the vertices moving always upon the rational norm-curve of the space under consideration.

EVANSTON,
*February* 4, 1905.

THOMAS F. HOLGATE,
*Secretary of the Section.*

---

## MATHEMATICS AT THE ST. LOUIS CONGRESS, SEPTEMBER 20, 22, AND 24, 1904.

IN the scheme of the Congress of Arts and Science connected with the Louisiana Purchase exposition, mathematics was classified as a department under the division of normative science, philosophy being the other department of that division. At 10 o'clock on Tuesday, September 20, both departments met and listened to the divisional address entitled "The science of the ideal," by Professor Josiah Royce, of Harvard University. An abstract of a paper itself so condensed as this is hardly a possibility; the paper has been published in *Science* (October 7, 1904).

Immediately following this joint session was the opening session of the department of mathematics, later to be subdivided into three sections. The officers of this session were Professor H. S. White, chairman, and Professor G. A. Bliss, secretary. Two addresses had been provided for this session, both of a highly general character, as distinguished from the more special discussions of sectional meetings. The first, by Professor Maxime Bôcher of Harvard University, was upon "The fundamental conceptions and methods of mathematics"; the second, by Professor James Pierpont of Yale University, on "The history of mathematics in the nineteenth century." Both have appeared already in this BULLETIN.* The auditors at this, as

* This volume, pages 115–135 and pages 136–159 respectively.

at the sectional meetings, numbered between 60 and 70.

On Thursday, September 22, the section of Algebra and Analysis met at 10 o'clock in Hall 1 of the library building, with Professor E. H. Moore as chairman and Professor G. A. Bliss as secretary. The two principal addresses were by Professor Emile Picard of Paris, entitled "Sur le développement de l'analyse mathématique et ses rapports avec quelques autres sciences," and by Professor Heinrich Maschke of Chicago, on "The present problems of algebra and analysis." The former has appeared in the *Bulletin des sciences mathématiques* for October and November, 1904, and will appear in an authorized translation in this BULLETIN within a short time.* The latter was an extended survey of the present state of the theory of invariants of quadratic differential forms in $n$ independent variables, an intensive study of differential parameters or the Biegungsinvarianten of surfaces, and included a resumé of the author's papers in the *Transactions* of this Society, Volumes 1 and 4.

Under the order of "Shorter Communications" the following papers were presented:

(1) E. V. HUNTINGTON: "A set of postulates and axioms for real algebra."

(2) J. I. HUTCHINSON: "On the uniformizing of algebraic functions."

(3) M. B. PORTER: "On functions defined by an infinite series of analytic functions of a complex variable."

(4) E. R. HEDRICK: "On the generalization of the analytic functions of a complex variable."

(5) M. W. HASKELL: "On collineations in ordinary space."

(6) J. B. SHAW: "On linear associative algebra."

(7) G. A. MILLER: "The bearing of several recent theorems in group theory."

In the absence of authors, No. 3 was read by Professor Bôcher, No. 7 by Professor Pierpont.

On Saturday at 10 o'clock, the section of Geometry met in the same hall, with Professor Thomas F. Holgate as secretary. In the absence of the chairman appointed by the directors of the congress, Professor W. E. Story, the section elected Professor M. W. Haskell as chairman. The two chief addresses

*A translation by Professor G. B. Halsted appeared in *Science*, December 23, 1904.

were by M. Gaston Darboux of Paris, "Etude sur le développement des méthodes géométriques," and by Dr. Edward Kasner of Columbia University: "The present problems of geometry." The former was published in the *Bulletin des sciences mathématiques* for September, 1904.* Dr. Kasner's address appeared in the March number of the BULLETIN.

At the close of the principal addresses, shorter communications were presented by the following:

(8) H. F. BLICHFELDT: "Concerning some geometrical properties of surfaces of revolution."

(9) G. A. BLISS: "The geometry of Jacobi's criterion for the problem of the calculus of variations involving several unknown functions."

(10) L. W. DOWLING: "On the generation of certain unicursal plane curves."

(11) ARNOLD EMCH: "The configuration of the points of inflexion of a plane cubic and their harmonic polars."

(12) G. B. HALSTED: "Non-euclidean spherics."

(13) H. HANCOCK: "Algebraic minimal surfaces."

(14) H. P. MANNING: "Representation of complex variables in space of four dimensions."

The third section, Applied Mathematics, met at 4 P. M. on Saturday in the gymnasium, with Professor A. G. Webster as chairman and Professor H. T. Eddy as secretary. The first address was on "The rôle of hypothesis in modern exact science", by Professor Ludwig Boltzmann, of Vienna; the second by Professor H. Poincaré, of Paris, on "The principles of physics." The latter has appeared, translated by Professor G. B. Halsted, in the *Monist* for January, 1905. Following these longer papers, Professor James McMahon presented a communication: "On the use of $n$-fold Riemann spaces in applied mathematics."

Aside from these strictly mathematical sessions, many meetings of cognate departments attracted hardly a less number of our members, and the publication of the collected papers from all departments will be eagerly awaited as the necessary adjunct and supplement to the oral proceedings of the Congress. One purpose of the directors of the Congress was certainly realized

*A translation by Professor Halsted appeared in the *Popular Science Monthly*, March, 1905.

— the conceptions of science as an organic whole, and of the community of interest, the necessary interdependence, of all special divisions of science were fitly expressed and strikingly enforced by this unique gathering. No narrow notion of a scientist's calling could survive for a day in that high atmosphere.

Brief outlines of the above papers, so far as they could be obtained, are given here. Papers already published are omitted. Numbers correspond to those in the lists above.

1. Dr. Huntington's paper has appeared in full in the *Transactions* for January, 1905.

3. Professor Osgood established in the *Annals of Mathematics*, volume 3 (1901), page 25, an important theorem on functions defined by a series of analytic functions. Professor Porter generalizes it by removing the condition that the point set shall be everywhere dense in the region $T$ of convergence, and substitutes the condition that the series shall converge for all values of the argument pertaining to a point set whose limiting points are everywhere dense on a closed rectifiable contour $C$ lying inside the region $T$.

4. Professor Hedrick proposed to generalize the notion of function $u + iv$ of a complex variable $x + iy$ by requiring the real and imaginary parts, $u$ and $v$, to satisfy not the Laplacian, but a more general form of linear differential equation of the second order. A specially interesting feature of these generalized functions would be the possibility that they might give a depiction of the $(x, y)$-plane conformal in the sense of an elliptic or hyperbolic geometry.

5. Professor Haskell announced several theorems on the resolution of collineations into generators of special type, and in particular on periodic collineations in space of $n$ dimensions.

6. Professor Shaw presented a topical outline of an extensive treatise, in which it is purposed to set forth the present state of linear associative algebra, not in a chronological or comparative study of the different forms of known algebras, nor in the exhaustive study of any one form, but in the tracing of the general laws of the whole subject. The paper is in three parts:

general theory, particular systems, and applications. The first part is thoroughly analyzed, consisting of twelve chapters developing the theory fully and independently, with a thirteenth which discusses the linear associative algebra as related to other algebras and to general algebra; of the second and third parts only a general description is given.

7. Professor Miller's short review referred mainly to enumerations of subgroups in groups of special type. A new theorem was the following. If a group of order $p^m$, $p$ being any prime, contains more than one cyclic subgroup of order $p^\beta > 4$, it contains a multiple of $p$ such subgroups, where $\beta$ again can have any value from 2 to $m - 1$.

8. If 4 or more points on any surface be taken, relations will subsist between their geodesic distances and the angles of the geodesics. Professor Blichfeldt proposes to exclude the angles from consideration, and inquire whether relations subsist among the distances. For more than 5 points, there will always be relations of this kind. For 5 points, if the surface be one of revolution, there will be one such relation. The author is not certain that the only surfaces having this property are those of revolution, but has proved that if two or more relations subsist among the 10 geodesic distances of 5 points on a surface, the surface has constant curvature.

10. In a net of plane curves with fixed fundamental points, the term involution curve is defined according to Castelnuovo. If an involution curve is composed of two algebraic curves, the two parts are termed *conjugate* in the involution defined by the net. Professor Dowling establishes certain general classes of degenerate involution curves having a line or a conic as one part, and so has a method of generating the conjugate part. This method he applies in particular to involutions of the second order, and obtains the the following three classes of unicursal curves: ($a$) a curve of order $2n - 1$ with multiple points indicated by $1_{2n-5}$, $(n - 3)_4$, $6_2$; ($b$) a curve of order $2n - 2$ with multiple points indicated by $4_{n-2}$, $((n - 3)/2)_4$, $3_2$; ($c$) a curve of order $2n - 2$ with multiple points indicated by $2_{n-1}$, $2_{n-3}$, $((n - 4)/2)_4$, $4_2$.

Using involutions of order higher than 2, curves usually not unicursal can be found as conjugates to particular lines and

conics. The author states for example that every plane quintic can be regarded as conjugate to a line in an involution determined by a net of cubics, having a suitable number of fundamental points.

11. By the aid of parametric representation of the points on a non-singular plane cubic, Professor Emch discusses the collinearity of the 27 points of contact of tangents to the cubic from its 9 inflexions. He finds 81 lines containing each 3 such points, and 81 other lines connecting 2 contact points with an inflexion.

12. Professor Halsted gave a brief outline of "intrinsic spherics", or spherical trigonometry independent of the parallel postulate, and advocated from a pedagogical standpoint the introduction of the sphere as two-dimensional figure into elementary geometry, instead of the use of the three-dimensional globe.

13. The problem of determining all algebraic minimal surfaces was solved analytically by Weierstrass in 1866 (see his collected works, volume 3, pages 39–52). Darboux, in his Théorie générale des surfaces, part 1, No. 221, has shown that such surfaces can always be generated by the translation of an algebraic curve. This shows that the function $F(s)$ used in the formulas of Weierstrass must be algebraic. Professor Hancock proposes as a problem next to be attacked the question of determining all those minimal surfaces which are not themselves algebraic, but contain a sheaf of algebraic curves.

H. S. WHITE.

EVANSTON, ILL.,
*March*, 1905.

---

## ON THE USE OF HYPERCOMPLEX NUMBERS IN CERTAIN PROBLEMS OF THE MODULAR GROUP.

BY DR. J. W. YOUNG.

(Read before the American Mathematical Society, September 17, 1904.)

THE following discussion connects two subjects which have hitherto been considered apart, and indicates a method of attacking certain modular group problems which seems susceptible of further elaboration.

By the modular group $\Gamma$, I mean the totality of substitutions $v(z) = (\alpha z + \beta)/(\gamma z + \delta)$, where $\alpha$, $\beta$, $\gamma$, $\delta$ are integers satisfying the relation $\alpha\delta - \beta\gamma = 1$. It is an old problem to define *arithmetically* the subgroups of $\Gamma$; except for the cyclic and so-called congruence subgroups it is as yet an unsolved problem.* In what follows I indicate a method of attack on such problems which seems not to have been employed hitherto. By applying this method in a special case I am led to the definition of the cyclic subgroups of $\Gamma$ and of congruence groups corresponding to cyclic quotient groups, the latter in an explicit form not before given.

The method referred to is as follows: Let $E$ be a commutative hypercomplex number system with a modulus and let $a$, $b$, $c$, $d$ be numbers of $E$ such that $ad - bc$ is neither zero nor a divisor of zero. Then the substitution $w(z) = (az + b)/(cz + d)$ has a unique inverse, $w^{-1}(z) = (dz - b)/(-cz + a)$.

If then for $v$ we place the totality of substitutions of $\Gamma$, the totality $w^{-1}vw$ will form a group simply isomorphic with $\Gamma$, whose coefficients are numbers of $E$. This may be called a hypercomplex representation of $\Gamma$, and it seems not unreasonable to suppose that a suitably chosen representation of this kind may bring to light relations very difficult to obtain with the ordinary representation.

To illustrate the method, I propose to apply it in a simple case, in which it leads in a direct and natural manner to the explicit arithmetic definitions of certain congruence subgroups.

Let $E$ be the two-unit system, whose units, $e_1$, $e_2$ combine as follows:

$$e_1^2 = e_1,\ e_1e_2 = e_2e_1 = e_2,\ e_2^2 = 0.$$

Also let $w$ be

$$w(z) = \frac{(e_1 + \frac{1}{2}ae_2)z + be_2}{ce_2z + e_1 - \frac{1}{2}ae_2},$$

whose determinant is $e_1$; $a$, $b$, $c$ being any real numbers. If we place

* By "arithmetic" definition is here meant one by means of an explicit arithmetic condition on the coefficients. Subgroups, neither cyclic nor of the congruence character, have been defined by conditions on the quotients in the continued fraction representation of the substitutions by Fricke, *Math. Ann.*, vol. 30 (1880), p. 345, and the author, *Trans. Am. Math. Soc.*, vol. 5 (1904), p. 88.

$$v(z) = \frac{az+\beta}{\gamma z+\delta}, \quad w^{-1}vw = v' = \frac{a'z+\beta'}{\gamma' z+\delta'},$$

we obtain after a short calculation

$$\alpha' = \alpha e_1 + [c\beta - b\gamma]e_2 \qquad \beta' = \beta e_1 - [a\beta - b(\alpha-\delta)]e_2$$

$$\gamma' = \gamma e_1 + [a\gamma - c(\alpha-\delta)]e_2 \quad \delta' = \delta e_1 - [c\beta - b\gamma]e_2.$$

Now it is clear that the totality of substitutions $v'$, in which the coefficients of $e_2$ are all zero, form a subgroup. In this way we find that the conditions

$$\text{(A)} \quad a\beta - b(\alpha-\delta) = 0, \quad a\gamma - c(\alpha-\delta) = 0, \quad c\beta - b\gamma = 0,$$

(of which the last is a result of the other two, if $a \neq 0$) define a subgroup of $\Gamma$, for any set $a, b, c$. It is easy to show that *the subgroup defined by* (A) *is always cyclic, and any complete* * *cyclic subgroup of* $\Gamma$ *can be defined in this way by a proper choice of* $a, b, c$. It hardly seems worth while to give the details of the proof.

Again, if $a, b, c, n$ are integers, it is evident that the totality of substitutions for which

$$\text{(B)} \quad a\beta - b(\alpha-\delta) \equiv 0, \quad a\gamma - c(\alpha-\delta) \equiv 0, \quad c\beta - b\gamma \equiv 0$$

form a subgroup. Suppose all the substitutions of $\Gamma$ to be reduced mod $n$, and denote the resulting group of finite order $\mu(n)$ by $G_{\mu(n)}$. In case $n$ is a prime $p$, it is easy to show that, for the different values of $a, b, c$, (B) gives the totality of cyclic subgroups of $G_{\frac{1}{2}p(p^2-1)}$.

The method of proof will be made manifest by considering in detail the cases in which $a \equiv 0 \bmod p$. We find then †

(i) If $b \not\equiv 0$, $B$ may be written

$$\alpha \equiv \delta, \quad \gamma \equiv \lambda\beta$$

where $b\lambda \equiv c$. The relation $\alpha\delta - \beta\gamma \equiv 1$, now becomes

$$\alpha^2 - \lambda\beta^2 \equiv 1.$$

Every solution of this congruence gives a substitution of the subgroup of $G_{\frac{1}{2}p(p^2-1)}$ in question; the number of incongruent

* By "complete" I mean not contained in any other cyclic subgroup.
† All congruences that follow are to be taken mod $p$.

solutions gives twice the order of the subgroup. Jordan * has given the number of solutions as follows: When $\lambda \not\equiv 0$, if $\lambda$ is a quadratic residue of $p$, there are just $p-1$ solutions; if $\lambda$ is a quadratic non-residue of $p$, there are just $p+1$ solutions. When $\lambda \equiv 0$, there are evidently just $2p$ solutions. For the different values $\lambda$, mod $p$, we obtain then

$$1 \text{ subgroup of order } p,$$
$$\tfrac{1}{2}(p-1) \text{ subgroups of order } \tfrac{1}{2}(p-1),$$
$$\tfrac{1}{2}(p-1) \text{ subgroups of order } \tfrac{1}{2}(p+1).$$

(ii) If $b \equiv 0$, $c \not\equiv 0$ we evidently obtain a single subgroup of order $p$.

The cases in which $a \not\equiv 0$ are treated similarly. By the transformation $\alpha = \tau + \nu$, $\delta = \tau - \nu$, the determination of the order and the number of the corresponding subgroups is made to depend on the same type of congruence as before. We find here $(a \not\equiv 0)$

$$p-1 \text{ subgroups of order } p,$$
$$\tfrac{1}{2}(p^2+1) \text{ subgroups of order } \tfrac{1}{2}(p-1),$$
$$\tfrac{1}{2}(p-1)^2 \text{ subgroups of order } \tfrac{1}{2}(p+1).$$

All the subgroups thus far obtained are clearly different, and by collecting results we have in all

$$p+1 \text{ subgroups of order } p,$$
$$\tfrac{1}{2}(p^2+1) + \tfrac{1}{2}(p-1) = \tfrac{1}{2}p(p+1) \text{ subgroups of order } \tfrac{1}{2}(p-1),$$
$$\tfrac{1}{2}(p-1)^2 + \tfrac{1}{2}(p-1) = \tfrac{1}{2}p(p-1) \text{ subgroups of order } \tfrac{1}{2}(p+1).$$

This by a well-known theorem† is the total number of subgroups of these orders contained in $G_{\frac{1}{2}p(p^2-1)}$, and they are all cyclic. The results may be stated as follows:

*The three pairs of congruences*

$$\left.\begin{matrix}\alpha \equiv \delta \\ \gamma \equiv \lambda\beta\end{matrix}\right\}, \quad \left.\begin{matrix}\alpha \equiv \delta \\ \beta \equiv 0\end{matrix}\right\}, \quad \left.\begin{matrix}\beta \equiv \mu(\alpha-\delta) \\ \gamma \equiv \nu(\alpha-\delta)\end{matrix}\right\}, \text{ mod } p \ [\lambda, \mu, \nu \text{ integers}]$$

* Traité des substitutions, p. 156.

† Klein-Fricke, Elliptische Modulfunctionen, vol. I, p. 434.

*define three mutually exclusive sets of subgroups, and these three together constitute the totality of the cyclic subgroups of* $G_{\frac{1}{2}p(p^2-1)}$. *Their number is seen to be* $p^2+p+1$. *The order of the subgroup is* $p$ *when* $\lambda \equiv 0$, *when* $4\mu\nu+1 \equiv 0$, *and in the second pair; the order is* $\frac{1}{2}(p-1)$ *when* $\lambda$ *or* $4\mu\nu+1$ *is a quadratic residue of* $p$; *the order is* $\frac{1}{2}(p+1)$ *when* $\lambda$ *or* $4\mu\nu+1$ *is a non-residue of* $p$.

This explicit form of definition seems to be new, and on account of its simplicity may be of interest.

NORTHWESTERN UNIVERSITY,
*September*, 1904.

---

## EXTENSION OF A THEOREM DUE TO SYLOW.

BY PROFESSOR G. A. MILLER.

(Read before Section A of the American Association for the Advancement of Science, Philadelphia, December 29, 1904.)

EVERY group $G$ of order $p^m$, $p$ being any prime number, contains at least $p$ invariant operators. This fundamental theorem, due to Sylow,* is included in the following: *Every non-abelian group of order* $p^m$ *contains at least* $p$ *invariant commutator operators, and its commutator quotient group* † *is always non-cyclic.* In this connection it seems desirable to prove the following closely related theorems: It is possible to construct a non-abelian group having any arbitrary abelian group as a commutator quotient group. Every non-cyclic abelian group of order $p^\alpha$ is the commutator quotient group of some non-abelian group of order $p^m$.

The first part of the theorem in italics may be proved as follows: Let $H$ represent the subgroup of $G$ which is composed of its $p^\beta$ invariant operators and let $H_1$ represent an invariant subgroup of order $p^{\beta+1}$ which includes $H$.‡ Any operator $s$ of $G$ which is not commutative with all the operators of $H_1$ transforms $H_1$ into a simple isomorphism with itself such that each of the operators of $H$ corresponds to itself. Since $H_1$ is

* Sylow, *Math. Annalen*, vol. 5 (1872), p. 584.

† It seems convenient to speak of the quotient group corresponding to the commutator subgroup as the commutator quotient group.

‡ Every invariant subgroup of a group of order $p^m$ is included in a larger invariant subgroup of arbitrary order less than $p^m$.

abelian, its remaining operators are transformed into themselves multiplied by operators of order $p$ contained in $H$.* These $p-1$ commutators of order $p$ are invariant under $G$ by hypothesis.

To prove that the commutator quotient group of $G$ is non-cyclic it is only necessary to observe that $G$ contains more than one subgroup of order $p^{m-1}$ † and that the operators which are common to all the subgroups of this order constitute a characteristic subgroup of $G$ with respect to which the quotient group is abelian and of type $(1, 1, 1, \cdots)$‡. As this quotient group has a $(1, k)$ isomorphism with the commutator quotient group,§ and is of composite order, the latter is non-cyclic. This completes the proof of the theorem in italics.

Since the commutator quotient group of the direct product of two groups is the direct product of the commutator quotient groups of these two groups, the proof that we can construct a non-abelian group whose commutator quotient group is the cyclic group of order $p^\alpha$, $\alpha$ being arbitrary, involves the proof of the theorem that it is possible to construct a non-abelian group whose commutator quotient group is an arbitrary abelian group. We proceed to prove this theorem.

It is well known that the infinite arithmetic series

$$1, 1+p^\alpha, 1+2p^\alpha, 1+3p^\alpha, \cdots$$

involves an infinite number of prime numbers. Let $p_1$ represent one of these primes. Since $p_1 - 1$ is divisible by $p^\alpha$ there are numbers which belong to exponent $p^\alpha$ mod $p_1$. In other words, there are groups whose commutator quotient group is the cyclic group of order $p^\alpha$. This proves the theorem in question. It may be remarked that every transitive substitution group of degree $p$ has a cyclic commutator quotient group.‖

The fact that every possible non-cyclic abelian group of order $p^\alpha$ is the commutator quotient group of some non-abelian group of order $p^m$ may be proved as follows: If the abelian group of order $p^\alpha$ has only two invariants $p^{\alpha_1}$, $p^{\alpha_2}(\alpha_1 \geqq \alpha_2)$, we

* BULLETIN, vol. 6 (1900), p. 337. *Annals of Mathematics*, ser. 2, vol. 3 (1902), p. 180.

† Burnside, Theory of groups, 1897, pp. 73, 75.

‡ Bauer, *Nouvelles Ann. de Math.*, vol. 19 (1900), p. 509.

§ *Quar. Jour. of Math.*, vol. 28 (1896), p. 268.

‖ BULLETIN, vol. 4 (1898), p. 141. Cf. Sylow, *Math. Annalen*, vol. 5 (1872), p. 594.

let $t$ represent any operator of order $p$ which transforms the generator $s$ of a cyclic group of order $p^{\alpha_1+1}$ into its $p^{\alpha_1}+1$ power. The commutator quotient group of the group generated by $s$ and $t$ is clearly of type $(\alpha_1, 1)$. Let $t'$ represent any operator of order $p^{\alpha_2}$ which is independent of $s$ and $t$. The operators $s$ and $tt'$ will then generate the required non-abelian group. By forming the direct product of this non-abelian group and some abelian group any additional invariants may be introduced into the commutator quotient group. Hence the theorem is proved.

STANFORD UNIVERSITY,
*December*, 1904.

---

## NOTE ON ISOTHERMAL CURVES AND ONE-PARAMETER GROUPS OF CONFORMAL TRANSFORMATIONS IN THE PLANE.

BY PROFESSOR C. L. BOUTON.

IN the January number of the BULLETIN, page 180, Mr. J. E. Wright integrated a certain differential equation by determining a continuous group which the equation admits. In solving the problem Mr. Wright determines a group of *conformal* transformations with *given* path curves. In this connection, it is an obvious problem to find the necessary and sufficient conditions under which a conformal group with given path curves shall exist. The solution of this problem is given in the following theorem:

THEOREM. — *A one-parameter group of conformal transformations with given path curves exists when and only when the given curves form an isothermal family.*

Although this theorem seems very obvious the writer cannot find it in print, and, therefore, gives two easy proofs for it.

I. Let $Uf = \xi p + \eta q$ be the symbol of the infinitesimal transformation of the group. Since the group is to be conformal, we must have $\xi + i\eta = \phi(x + iy)$. The differential equation of the path curves is $\eta dx - \xi dy = 0$. From this equation we have

$$\frac{dx + idy}{\xi + i\eta} = \frac{dx - idy}{\xi - i\eta},$$

or

$$\frac{d(x+iy)}{\phi(x+iy)} = \frac{d(x-iy)}{\phi(x-iy)}. \tag{1}$$

Let

$$\int \frac{d(x+iy)}{\phi(x+iy)} = u(x, y) + iv(x, y).$$

We then have, on integrating (1), $v(x, y) =$ constant, which is the equation of the path curves. But $v(x, y)$ is a solution of Laplace's equation, hence the path curves of any conformal group are isothermal curves. Moreover, if any solution of Laplace's equation be given, we may use this as $v(x, y)$ to determine $\phi$, and we see at once that $\xi$ and $\eta$ are uniquely determined, save as to a multiplicative constant. That is, given a family of isothermal curves, there is one and only one group of conformal transformations which has these curves as path curves.

II. Let us assume that the path curves are given as the integral curves of the differential equation $y' = F(x, y)$. Then we must have $\xi p + \eta q \equiv \rho(x, y)[p + Fq]$. Or $\xi \equiv \rho, \eta \equiv \rho F$. The conditions $\xi_x = \eta_y$, $\xi_y = -\eta_x$ give

$$\frac{\partial \log \rho}{\partial x} - F\frac{\partial \log \rho}{\partial y} = F_y,$$

$$F\frac{\partial \log \rho}{\partial x} + \frac{\partial \log \rho}{\partial y} = -F_x.$$

Solving these equations, we have

$$\frac{\partial \log \rho}{\partial x} = \frac{F_y - FF_x}{1+F^2}, \qquad \frac{\partial \log \rho}{\partial y} = \frac{-F_x - FF_y}{1+F^2}.$$

Applying the condition of integrability to these equations, we find as the necessary and sufficient condition that $\rho$ shall exist

$$(1+F^2)(F_{xx}+F_{yy}) - 2F(F_x^2 + F_y^2) = 0. \tag{2}$$

But this is the necessary and sufficient condition that $y' = F(x, y)$ shall have as its integral curves a family of isothermal curves. * If this condition is satisfied the value of $\rho$ is given

* See Lie-Scheffers, Differentialgleichungen, p. 157; Kasner, BULLETIN, vol. 10 (1903–04), p. 342.

by the equation

$$\rho = e^{\int \frac{F_y dx - F_x dy}{1+F^2}} / \sqrt{1+F^2},$$

and is determined save as to a multiplicative constant. The group is therefore uniquely determined.

In the second proof it is assumed that the meaning of the condition (2) is known. If, however, the condition (2) had not been derived independently, the two proofs together show that (2) is the necessary and sufficient condition that the integral curves of $y' = F(x, y)$ shall form a system of isothermal curves.

HARVARD UNIVERSITY,
*February*, 1905.

---

## ARENDT'S DIRICHLET'S DEFINITE INTEGRALS.

*G. Lejeune Dirichlet's Vorlesungen über die Lehre von den einfachen und mehrfachen bestimmten Integralen.* Herausgegeben von G. ARENDT. Braunschweig, Vieweg und Sohn, 1904. xxiii + 478 pp.

This book is almost a literal reproduction of the course on definite integrals which Dirichlet gave at Berlin during the summer of 1854. It is not its aim to give any account of the development of the subject during the last fifty years. The book on definite integrals by Meyer * contains discussions of trigonometric series, potential and other matters, taken partly from other courses of Dirichlet, and partly from his own investigations. Whether the new book encroaches on the older one is not necessary to discuss, for Meyer has long been out of print and it is certainly worth while to have the Dirichlet course accessible, essentially in the form in which it was given. Moreover, apart from the questions of continuity, integrability, length, area, uniform convergence, etc., the great body of subject matter is to-day what it was then.

After defining continuity, an integral is discussed by means of a figure which illustrates the area included between two ordinates, the axis of $X$ and a continuous curve. The same problem is then treated analytically, for an arbitrary division of the in-

* Vorlesungen über die Theorie der bestimmten Integrale zwischen reellen Grenzen, mit vorzüglicher Berücksichtigung der von P. Gustav Lejeune-Dirichlet im Sommer 1858 gehaltenen Vorträge über bestimmte Integrale. Von Dr. Phil. Gustav Férdinand Meyer. Leipzig, Teubner, 1871.

terval of integration. After the usual properties of finite integrals are derived, the ideas are extended to apply to isolated points of discontinuity (poles) and to integrations extending over an infinite interval. It must be remembered that the course was given before Riemann's papers appeared. The next chapter is devoted to the integration of rational fractions between infinite limits; the treatment is fuller than that given by Meyer, and is followed by numerous applications. A similar detailed discussion is given to the integral of $\cos x^2$ between infinite limits. Arendt points out that Dirichlet used the term theorem of mean value at least five years before Paul du Bois Reymond's dissertation appeared.*

The remainder of the first part (pages 95–222) is devoted to the eulerian integrals. The treatment differs materially from that in the corresponding part of Meyer's book. No use is made of the series for numerical calculation, the aim being to reduce a large number of apparently different types of integrals to depend upon gamma functions. Free use is made of imaginaries as parameters, rather than as complex variables. The matter is prefaced by a digression of about twenty pages on fundamental theorems concerning functions of a complex argument, and the continuity of a definite integral with regard to a parameter. Necessary conditions for the uniformity of inverse functions are discussed both analytically and graphically.

The special types considered at length are

$$\int_0^\infty e^{-(\kappa+\theta i)x}\, x^{a-1}\, dx = \Gamma(a)\,(\kappa+\theta i)^{-a} \qquad (\kappa > 0)$$

and $\int_{-\infty}^{\infty} \frac{e^{xi}}{(\kappa+ix)^a}\, dx$, including a large number of particular cases, and finally the discontinuous integrals of the type $\int_0^\infty \frac{\sin x \cdot \cos hx}{x}\, dx$. The subject treated by Meyer in chapter 4 (pages 355–435) is omitted. The development is remarkably clear; it is presented in an elementary way that makes very easy reading.

* Compare Kronecker's Vorlesungen über die Theorie der einfachen und der vielfachen Integrale. Leipzig, Teubner, 1904, page 78.

The applications of simple definite integrals were reserved for the weekly "Uebungen" which accompanied the course. Those considered are the summation of harmonic series; the relation between general factorials and the Eulerian integral of the first kind* and their limiting values for an infinite argument; infinite product representation of $\sin \pi x$ (without the exponential factor to insure convergence); Taylor's series including the remainder term; development and discussion of the exponential series, and the limit of $\Gamma(n)$ for large values of $n$; convergence of the hypergeometic series, including points at the ends of the (real) interval of convergence; finally, the fundamental properties of complex imaginary functions.

The second part is concerned with multiple integrals. It begins with the transformation of the variables, and the application of double integrals to finding the area included within plane curves and of curved surfaces. It is pointed out that the statement that the straight line is the shortest distance between two points needs to be proved, and that it has no meaning to speak of a general curved surface being flattened out. Length and area as applied to curved lines and surfaces are sharply defined before the formulas are applied. A paucity of $\epsilon$ proofs is apparent, and free use is made of space intuition. This is particularly the case in the discussion of the change of sequence of integration between variable limits. The transformation of the variables in double integrals occupies ten pages. This subject is presented in an exceptionally clear and convincing manner. A very detailed application is given to the problem of finding the area included within the ellipse, both in rectangular and polar coördinates, with each sequence of integration. Curvilinear coördinates are introduced to determine the area of curved surfaces. Thirty pages are given to the problem of finding the area of the general ellipsoid by various methods.

Thus far but few historical references are given, and when names are mentioned, it is only in a general way, without giving exact citations. In a book whose merit is mainly pedagogical this procedure is justified, particularly because the proofs given by Dirichlet differ in essential features from those of his predecessors.

The chapter of ninety pages on triple integrals is devoted

*No reference is made to Binet's memoir, nor are these functions named $B(a, b)$. The notation used is $(a, b)$. They receive much less attention than is given them in Meyer's book.

almost entirely to the problem of the attraction of an ellipsoid. A historical introduction is followed by the expression for a space element in polar coördinates, then comes the statement of the problem concerned, first for any law of attraction, then for the Newton law of gravitation to which most of the subsequent development is confined. The first case considered is that in which the attracted point lies within the ellipsoid. The density, mass, and constant factor of attraction are all assumed to be unity. The axes are transformed to parallel ones through the point, then polar coördinates are introduced, the limits of integration for each variable being carefully determined. From the form in which the constants enter the final integrand the usual theorems are derived; the integral itself is reduced to the elliptic type, then this case is left. Besides the historical references given in the text, the appendix contains exact citations to an extensive literature on the subject.

The case in which the attracted point is outside the ellipsoid is next taken up. It is first shown wherein the difficulty of applying the preceding method lies, then Ivory's theorem is brought in and shown to apply to every case. This discussion, applied to a system of confocal ellipsoids, establishes the fact that the problem is essentially the same for every ellipsoid of the system, hence from this standpoint, the case of the external point can be reduced to that of an internal point by changing the constants.

Finally, the same problem is discussed by means of the discontinuous factor. The expression

$$\frac{2}{\pi}\int_0^\infty \frac{\sin\phi}{\phi}\cos\kappa\phi\, d\phi$$

has the value 1 when $-1<\kappa<1$, the value 0 when $\kappa<-1$ or $>1$, and the value $\frac{1}{2}$ when $\kappa=\pm 1$. If the coördinates $x, y, z$ of a point within the ellipsoid

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$$

be substituted in its equation, the first member has a value less than unity; if the point be on the surface, the first member is unity, and if the point be outside, it is greater than unity. Hence the integral

$$\frac{2}{\pi}\int_0^\infty \frac{\sin\phi}{\phi}\cos\left(\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}\right)\phi\,d\phi$$

has the constant value unity for every point within the ellipsoid; it becomes zero for points outside and has the value one half for points on the surface. The potential of the ellipse is now employed, and imaginary parameters are introduced, by means of which the integrals are readily reduced to elliptic forms, similar to those obtained in the preceding chapter.

In the last chapter the discontinuous factor is employed to reduce an extended type of multiple integrals to gamma functions. The results are applied to the determination of volumes and moments of inertia of certain solids. The substance of this chapter was published in *Liouville's Journal* in 1839, but has been materially improved in its present form.

In a series of notes appended to each part every slight variation in the text from the form originally given by Dirichlet is pointed out. Occasionally further steps of a proof are there reproduced, which are now known to be unnecessary. On the other hand, many earlier authors had contented themselves with simpler proofs which in many cases are shown to be insufficient.

Typographically, the book is uniform with the well known Theory of numbers of Dirichlet-Dedekind, and the books of H. Weber, published by the same house. Apart from the errors which are mentioned at the end of the volume, the book is remarkably free from typographical errors.

VIRGIL SNYDER.

---

## THE THETA FUNCTIONS.

*Lehrbuch der Thetafunktionen.* Von ADOLF KRAZER. Leipzig, Teubner, 1903. 8vo. xxiv + 509 pp.

*Theorie der Riemann'schen Thetafunction.* Von GEORG ROST. Leipzig, Teubner, 1901. 4to. iv + 66 pp.

The masterly treatise of Professor Krazer provides the mathematical public for the first time with an adequate and satisfactory account of the theta functions. These functions being the elements out of which uniform periodic functions of any number of variables, without finite essential singularities, may be constructed, they afford the most powerful instrument

for the investigation of such periodic functions and suffice for a complete theory of them. Periodic functions of the kind specified, including as they do the elliptic, hyperelliptic, and abelian functions, undoubtedly constitute the most important class of special functions at present known.

The sources of information on the theta functions hitherto most readily accessible have been some very excellent chapters included in various books on general or particular function-theory, together with three small volumes containing investigations by Krazer and Prym which have been the direct precursors of the present work. The most recent of these, the Neue Grundlagen einer Theorie der allgemeinen Thetafunktionen, a splendid piece of work in the remodeling of the general theory and extending the treatment to functions with rational characteristics, has for more than a decade done good service in lieu of a text or reference book.

The delay in the appearance of a suitable text in this important subject is compensated for by the admirable treatise now before us. It is in every respect a model. The author has devoted his life to this particular field, and no one is better qualified to write upon it than he. A vast amount of material has been systematized and condensed into a moderate compass. Many of the details, especially those relating to the transformation theory, are of necessity quite intricate, but the entire theory has been presented in such a lucid and attractive form, and the formulas are derived with such directness, simplicity, and absence of artifice, that the efforts of the reader are reduced to a minimum. The reader is materially aided, too, by the remarkable freedom from typographical errors. There are over 2,100 numbered formulas in the book, many of the numbers representing several equations. The reviewer has verified the greater part of these, and has found only the following corrections needed.

On page 4 insert the term $+ k/m$ in the exponent of the last member of (10); on page 8 the last $A$ in (23) should have the subscript 0; on page 36, fourth line from the end of §5, read (XXXIII) instead of (XXIII); on page 41 the first $g$ in (121) should have an accent; on page 154 read

$$B = (N^2M^2NM)^3$$

in (110); on page 164 insert $g$ in the characteristic symbol

in (136); on page 165 the $c$ after the first summation sign in (141) should be raised above the level of its pair of subscripts; on page 173 the last $x$ in (163) should have the subscript $\nu$; on page 180 the factor $G$ [cf. (XXXI) and (XXXII) page 181] should appear in the right member of (203); on page 254 interchange "ungeraden" and "geraden" in lines 4 and 5 from the top; on pages 502–3, insert 485, 356, 356, 356, 125 in the references to Biermann, Krazer, Noether, Prym, Wirtinger respectively; on page 502 change 362 to 361 in the reference to Cayley; on page 503 change 23 to 22 in the references to Riemann and Weierstrass.

We proceed to give a brief outline of the subject matter.

The first chapter treats of the $p$-fold theta series with rational characteristics, and its functional properties.

The next two chapters deal with the general problem of transforming a $q$-fold infinite series by means of a linear substitution, with rational coefficients, of the $q$ letters of summation. The formulas so obtained are applied to the derivation of a number of relations between theta functions with different arguments and moduli.

The fourth chapter deals with the properties of uniform periodic functions without finite essential singularities, and gives a proof that such functions can always be expressed rationally in terms of theta functions. The complete proof of this theorem, announced by Riemann in 1860, has only recently been obtained. The chief difficulty has been to show that the $2p$ systems of periods of any such periodic function in $p$ variables must always satisfy conditions equivalent to those to which the theta moduli are subject.

On the other hand the abelian functions were at first supposed to form a special class of periodic functions, since such functions of genus $p > 3$ depend on fewer parameters than the general periodic function. That the one class is coextensive with the other is proved in chapter XI, which is devoted to a discussion of the conditions for the reducibility of abelian integrals and the associated theta functions. It is shown that special classes of abelian functions of genus $q > p$ can be found whose associated theta functions break up, after a suitable transformation, into products of two theta functions, one of $p$ and the other of $q - p$ variables, of which the first is a theta function of the most general kind.

Chapter V deals with the most difficult part of the subject,

the transformation of the theta functions. It is shown how every transformation may be compounded of certain linear and non-linear transformations. The general formula for a linear transformation in which the determinant of the coefficients $c_{\mu,\nu+p}(\mu, \nu = 1, 2, \cdots, p)$ occupying the second quadrant is not zero, is completely worked out. A transformation in which this determinant is zero is later shown to be expressible as a combination of transformations of the preceding kind.

Chapter VI deals with the complex multiplication of the theta functions.

Chapter VII, containing 130 pages, treats in detail of functions with half-integer characteristics. It deals at some length with Frobenius's theory of groups and systems of characteristics. The Riemann theta formula and various relations obtainable from it are next derived, following which the special formulas for the cases $p = 1, 2$ are deduced and then the addition theorem for $p > 2$, closing with a general discussion of theta relations.

The two succeeding chapters give the familiar Riemann theory of the abelian and hyperelliptic ($p \geqq 2$) theta functions.

The author's systematic method of fully defining all the symbols used in the enunciation of each theorem greatly enhances the value of the book for reference purposes. The historical and bibliographical remarks are given in smaller type at the ends of the sections to which they relate. They are very full and suggestive, and add greatly to the value and completeness of the work.

Two indexes are given, one of the authors quoted and another of the subject matter, while the fourteen-page table of contents furnishes quite a minute analysis and a systematic résumé of the volume.

The Habilitationsschrift of Dr. Rost is a critical revision of the Riemann theory, having for its leading aim to close certain gaps which have heretofore existed. The memoir is divided into two parts of equal length. The first part is preparatory to the second and gives a clear and systematic account of algebraic functions, with especial reference to properties of point systems on the Riemann surface which can serve as infinites for such functions. There is nothing new in this portion of the subject except the form in which it is presented.

The second part gives a complete discussion of the theta function regarded from Riemann's point of view as a function

of position in the Riemann surface. The interest centers in the point systems at which the theta function vanishes. To complete the theory in certain respects, in which it is still open to criticism on account of the possibility of the occurrence of point systems of special character, is the chief aim of this part of the paper. This, together with the critical remarks and illustrative examples collected into an appendix of four pages at the end of the work, form a new and useful contribution to the Riemann theory.

J. I. HUTCHINSON.

CORNELL UNIVERSITY.

---

## MATHEMATICAL CRYSTALLOGRAPHY.

*Mathematical Crystallography and the Theory of Groups of Movements.* By HAROLD HILTON. Oxford, The Clarendon Press, 1903. 262 pp.

THE problem of crystallography is to establish a correspondence between chemical composition or certain abstract aspects of it and geometric form in crystals.

Setting aside the work, usually assigned to the mineralogist, of reducing the actual forms of crystals to idealized representatives, and the chemist's problem, of establishing the molecular configuration, and showing why given molecules are more likely to be piled in one way than another, the subject matter of mathematical crystallography is the formal problem of showing a correspondence between the idealized forms of the mineralogist and the geometrically possible methods of piling, that is of filling space of three dimensions with interpenetrant homogeneous assemblages. Mr. Hilton's work undertakes to set this out as it has been done by Bravais, Sohncke, Schoenflies and others, and also to supply such geometric material as may be pertinent.

Dealing first with the main issue the definition of subject matter (page 11) amounts to this: The idealized forms of natural crystals are polyhedra with rational indices; more explicitly polyhedra whose faces are parallel to the set of planes

$$\frac{hx}{a}+\frac{ky}{b}+\frac{lz}{c}=0,$$

where $a, b, c$ are any real positive quantities and $h, k, l$, the "indices," are integers, usually small.

This is a sufficient definition and does not need combination (page 11) with the law of constancy of angles, which is a consequence unless $a:b:c$ vary, as in calcite with the temperature.

The formulas given (page 20) for calculation of indices involve four independent angular measurements in a manner invariant under this expansion.

The line of argument is then briefly:

1. Each of the rational index polyhedra ($R$) is invariant under some point-group ($P$) of a set of 32. Point-groups are rotation groups or extended rotation groups.

2. There exists a set of 14 lattices, or homogeneous assemblages of points ($A$), each one invariant under a translation group $T$ containing 3 independent finite translations and no infinitesimal translation.

3. Each lattice $A$ is invariant under some $P$ and only under point groups of the set $P$. There exist 230 space-groups ($S$) which contain some $T$ as a subgroup and some $P$ as a maximum subgroup of the class.

4. Every system of interpenetrant homogeneous assemblages ($M$) is invariant under some $S$ and only under space-groups of the set $S$.

The crystallographer and chemist are then left with the problem of determining, for a given crystal of geometric form $R$ and given chemical and physical properties, which of the sets $S$ and $M$ shall be assigned to it.

To establish the invariance of $R$ under $P$ it is proved (Chapter III) that symmetry axes only of orders 2, 3, 4, 6, occur in $R$. There is trouble and apparently avoidable trouble with the proof. As a lemma it is proved (page 40) that a symmetry axis is perpendicular to a possible (*i. e.*, rational index) face. This does not hold for three-fold axes as is carefully pointed out. Still a corollary of this lemma is used (§ 19) to prove $\cos 2\pi/n$ rational, whence $2\cos 2\pi/n$ must be integral. The conclusion $n = 2, 3, 4$ or 6 ignores the formal solution $n = \infty$, and of course the chain of reasoning for $n = 3$ is hopeless. The exclusion of 5, 7, 8, etc., and the admission of 2, 3, 4, 6, which is the object of the discussion, can be accomplished however, and the appeal to the structure theory (page 39) is irrelevant.

The enumeration of ($P$) the point groups having only 2-al, 3-al, 4-al, and 6-al symmetry axes is given in Chapters V and VI. After a short introduction to the theory of groups the

method is direct and involves the metrical element in the way of spherical areas. Hence Euler's theorem, one of analysis situs applicable to a much wider class of objects than those discussed, is said rather strangely to be *deduced* (page 69).

Two proofs somewhat similar are given (§ 4 and § 5). In the former the angle between axes is assumed acute (page 56). This excludes the tetrahedron group as generated by 3-al rotations, yet the case comes back in the solution of an inequality in integers to be thrown out again on the ground of the obtuse angle. In the second proof the spherical triangle formed by vertices of the inscribed tetrahedron is specifically excluded. The tetrahedron, however, turns up in the table (page 59) to be omitted in the subsequent discussion until its group is reinstalled as a subgroup of the octahedral. As a crystal the tetrahedron is a tetartohedral form, its group being a subgroup of index 4 under the extended octahedral group, but there seems no reason to insist on this at this stage.

The rotation groups are extended by an operation of the "second sort" in Chapter VI. The diagrams, stereographic projections, are clear and satisfactory.

The general theory of rows, nets and lattices is given in Chapter XIII and the possible symmetry of lattices discussed in the following chapter. This is done clearly and well.

The enumeration of the 230 space-groups necessarily involves a rather lengthy statement (Chapters XVIII to XXIII), and though it might be possible and desirable to arrive at the number and a bare classification with less labor, yet from the point of view of aiding definite work in placing actual crystals in correspondence with the different space-groups the more detailed work should stand. The figures, of course rather symbolic, supply an adequate representation.

In addition to the special subject the book contains an exposition of various pure geometrical theorems connected with it. The theory of stereographic projection claims Chapter I. Here it might be noted that the lemma of § 4 can be proved by symmetry without so direct an appeal to the congruence of triangles. If the student needs this the use of "perpendicular" to describe a relation between non-coplanar lines (page 18) errs in excessive brevity.

The analytic geometry of rotation as treated in Chapter VII uses a notation involving the cosines of angles written explicitly. In Note I to the chapter the author calls attention to the

greater neatness of the direction ratio notation but uses only a single subscript. Note II introduces a double subscript notation but only to prove an elementary determinant identity by the construction of a rather unwieldy polygon in three dimensions.

The discussion of the properties of geometric operations in Chapter XVI makes good use of the group theory and is very satisfactory.

The suggested practical method of finding the resultant of several screws involves the determination of the first and last positions of some "easily ascertained point." Of course, there may be, and in the application to crystallography (where the screw angles are limited to a few aliquot parts of the complete rotation) usually are such points. In general, however, all points are about in the same case.

The exposition of the group theory here required is rather abbreviated but satisfactory, except the definition of "isomorphous" (page 158). This word, usually kept for the chemical connotation, replaces practically the word "isomorphic" as used in group theory as far as it applies to this subject. But the restriction in the definition is hardly necessary.

The proposition (page 159) that "the symmetry of lattice is in general the holohedry of which a point group under which it is invariant is the merohedry" seems to use the words "in general" to mean in a majority of a finite number of cases. There seems to be no general theorem applicable, since the quotient groups of two groups may be related as group and subgroup without the original groups being so related, whenever the denominators in the quotient symbols differ.

The necessary references to the historical and physical side of the subject are satisfactory. The definition of crystalline (page 8) seems to be that usually given for eolotropic. Do there not exist outside the archimedean universe we are here dealing with "bodies" properly called eolotropic and yet not crystalline? The uniformly magnetized bar is not usually classed as crystalline; its geometric ideal has a symmetry axis of infinite order, the case ignored by the author.

The whole subject is one of extreme interest to the physicist and chemist, and the mathematics involved, though rather special, constitute a good field for the exercise of geometric powers.

Thanks are due to the author for his labors in gathering these things together and setting them forth in a connected way.

R. P. Baker.

## THE THEORY OF ELECTRICITY.

*Theorie der Elektrizität.* Von M. ABRAHAM. Erster Band: *Einführung in die Maxwellsche Theorie der Elektrizität.* Von A. FÖPPL. Zweite, vollständig umgearbeitete Auflage. Von M. ABRAHAM. Leipzig, B. G. Teubner, 1904. xviii + 443 pp.

BEFORE this, we have had occasion to praise Föppl's lectures on technical mechanics.* His treatise on electricity is no less deserving. When this first appeared in 1894 it was, at all events for Germany, a new departure. Vector analysis was consistently employed throughout, and the point of view of Maxwell, which was so long in winning its proper recognition on the continent, was unequivocally and wholly adopted, even for elementary instruction. It must be a cause of gratification to Föppl to find a new edition called for in less than a decade; and to us it should be a matter for rejoicing that the new edition is not merely an improvement on the old, but an extension of it so as to form in two volumes a really modern treatise on electricity and magnetism from the Maxwell-J. J. Thomson point of view.

The theory of electrons and the atomic nature of electricity have recently become so firmly established as a highly valuable, if not yet indispensable, interpretation of nature, that a connected treatise beginning with the equations of Maxwell and leading through the motion of charged bodies to the consideration of atomic electricity and electrical mass is justified and necessary. Undoubtedly the present work will contribute to the popularization and ultimate adoption of the points of view for which it stands. Abraham, well known for his researches along just these lines, is peculiarly fitted to undertake this revision and enlargement of Föppl's treatise. The task has not been small. It has been ably executed.

The first section, containing the theory of vectors and vector fields, has been so amplified as to give the reader a summary exposition of those theorems in dynamics and hydrodynamics which form the basis of present views of electrical action. Surfaces of discontinuity, divergence, curl, and their relation to the theory of double layers (Doppelschichten, couches doubles) have their due attention. It is pointed out that the divergence and curl of a vector function $\mathbf{V}$ are given by the formulas

* Reviewed in this BULLETIN, vol. 9, pages 25–35.

$$\operatorname{div} \mathbf{V} = \lim_{d\tau \doteq 0} \frac{\int\int_s d\mathbf{S} \cdot \mathbf{V}}{d\tau}, \quad \mathbf{n} \cdot \operatorname{curl} \mathbf{V} = \lim_{dS \doteq 0} \frac{\int_c d\mathbf{r} \cdot \mathbf{V}}{dS},$$

where $d\tau$, $d\mathbf{S}$, $d\mathbf{r}$ are respectively the elements of volume, of surface, and of arc, and where $\mathbf{n}$ is a unit vector in the direction $d\mathbf{S}$. It may be observed that a very fruitful method of introducing the divergence and curl is obtained by adopting these expressions as definitions. It is not difficult to show that for ordinary vector functions these limits exist independently of the way in which $d\tau$ and $d\mathbf{S}$ approach zero — or these facts may be taken for granted without introducing any more assumptions than are frequently made in disregarding the undesired parts of integrals arising in connection with the theorems of Gauss and Stokes.

In regard to notation Föppl essentially followed Heaviside.* Abraham follows the usual convention, and adopts a new notation of his own. In the Encyclopedia,† he used italic type for vectors and denoted the components by subscript letters — $A$, $A_x$, $A_y$, $A_z$. Föppl originally used heavy German type for vectors and the corresponding letters with subscript numbers for the components — $\mathfrak{A}$, $A_1$, $A_2$, $A_3$. He used $\mathfrak{AB}$ and $V\mathfrak{AB}$ for the scalar and vector products. Lorentz in the Encyclopedia ‡ used ordinary German type for vectors and the same with subscript $x$, $y$, $z$ for the components. He employs $\mathfrak{AB}$ and $[\mathfrak{AB}]$ for the scalar and vector products. Abraham now uses heavy German type for vectors and the same with subscript letters for the components — $\mathfrak{A}$, $\mathfrak{A}_x$, $\mathfrak{A}_y$, $\mathfrak{A}_z$. This prolific use of heavy type, even for the components which are scalars, produces an ugly looking page. For the scalar and vector products he uses the same notation as Lorentz. It might have been as well to retain Heaviside's notation. This would facilitate reading Heaviside; and reading Heaviside must be one of the aims of every student of electromagnetic theory. But evidently we do not approach a uniform notation as time goes on and new works appear.

In the main body of the work the present editor has rewritten and rearranged to a considerable extent, showing frequently

* "Electrical papers" (1892) and "Electromagnetic theory" (1893, 1899).

† Volume 4, part 3, pages 1–47 (1901).

‡ Volume 5, part 2, pages 63–280 (1904).

more the influence of J. J. Thomson* and Lorentz, where the original author inclined rather to Heaviside. Thus Abraham writes

$$\mathbf{I} = \frac{\mu - 1}{4\pi}\mathbf{H},$$

where Föppl wrote

$$\mathbf{I} = \frac{\mu - \mu_0}{4\pi\mu_0}\mathbf{H};$$

and $\mu = 1 + 4\pi\kappa$ where the original had $\mu = \mu_0(1 + 4\pi\kappa)$. Heaviside and Gray† would probably think this a step backward — and so indeed it is when viewed from the theory of dimensions. For $\kappa$ is a pure number and not of the doubtful dimensions of $\mu$. To be sure, the matter is carefully explained in the text, where it is pointed out that the dimensions of $\epsilon$, $\mu$, and $c$ are chosen differently in different systems of units; and that in the system which is here adopted $[\epsilon] = 0$, $[\mu] = 0$, and $[c] = LT^{-1}$. Nevertheless in a work which is for the most part theoretical and where computation in any particular system is generally foregone it would seem wiser not to specialize the dimensions in this manner. The world over, seconds are used as units of time. Time might therefore be assumed as a pure scalar devoid of dimensions — but mechanics would thereby lose. Although we might not agree with Heaviside that one system of units is more "rational" than another (it may be more symmetrical, but irrationality is bound to come in if one has squares and circles to measure), we are inclined to agree with him in the matter of dimensions and to regret the alteration of Föppl equations to a form in which they are good only for one particular system of units.‡

To one acquainted with recent English and American works on electricity it appears very strange to read through a considable volume without finding much if anything concerning Faraday tubes. The reason for the omission is not far to seek. If one is familiar with the underlying conceptions of dynamics, and

*We may note J. J. Thomson's "Elements of Electricity and Magnetism" just out in a third edition. It covers largely the same ground as the book under review.

†Treatise on Magnetism and Electricity," Volume 1 (1898).

‡The whole matter of systems of units for electromagnetic theory seems unfortunately becoming more and more complicated and diversified, as is the case with vector methods. More than half a dozen systems are set forth by Lorentz, *loc. cit.*

especially of hydrodynamics as set forth in the earlier chapters of the book, this Faraday tube is not a crying necessity. It offers a good visual image of a number of things going on in the medium — tension along the lines of force and pressure perpendicular to them. It may even serve to give a good idea of what takes place in the medium when a condenser discharges. And we should have been glad to see some exposition of the Faraday tube by Abraham and Föppl. But it is better to leave this mechanism or model or visualization — whatever one wishes to call it — untouched than to push it so far as to leave the reader with too crudely exaggerated a notion of its real existence if it is but a visualization, or with a conception which the reader cannot push further without bewilderment in case the tube is a reality.

The statement is often seen that from each negative electric atom there issues one unit Faraday tube terminating upon some surface of positive charge. We have been asked by physicists of considerable eminence whether the electron dragged its tube after it, or pushed it in front, or how the thing *did* look in motion or at rest. And small wonder, when the chief authority writes:* "The Faraday tubes stretching through the ether cannot be regarded as entirely filling it. They are rather to be looked upon as discrete threads embedded in a continuous ether, giving to the latter a fibrous structure; but if this is the case, then, on the view we have taken of a wave of light, the wave itself must have structure, and the front of the wave instead of being, as it were, uniformly illuminated, will be represented by a series of bright specks on a dark ground, the bright specks corresponding to the places where the Faraday tubes cut the wave front."

We are not prepared to dispute Thomson. His piercing imagination and scientific method have rendered too great services to physics. We are simply unable to follow him. An ether quasilabile or rigid, and atoms of electricity positive and negative, and Faraday tubes of the old-fashioned hydrodynamic sort we readily admit; but that the harmony of the spheres should be due to the transverse vibrations of myriads of tightly stretched strings (how long, and stretched between what?) embedded in an ether is a little more than we are yet ready to comprehend. One is tempted to feel that the harmony would

* J. J. Thomson, "Electricity and matter," page 63 (1904).

become a terrible and discordant jangle after a few reflections, interferences, rotations, and the like. Certainly the introduction of this additional stringiness adds a considerable complication to our conception of the ether, which for the most part has always been regarded as homogeneous except as to points (matter) rather than lines (Faraday tubes). There seem to be, however, phenomena † which this picture explains. Perhaps the universe does indeed look like a tremendous boiling pot of spaghetti. On this point Föppl and Abraham are silent.

After thus developing at some length a few points which the Theorie der Elektrizität has suggested, we shall not take away the zest of the reader by entering into details of the work under review. It remains to state in closing that in the revision the modern point of view is so fully adopted and the connections with the older theories so little discussed that one who had been brought up on the old would have some difficulty in immediately comprehending the new, notwithstanding its logical excellence and clear exposition. But the utility of modulations has largely vanished during the past decade, and Abraham is undoubtedly right in omitting them. This book is written by the young school for the young school — and we wish it an increase of its deserved success. The second volume will be awaited with impatience.

EDWIN BIDWELL WILSON.

YALE UNIVERSITY,
*February*, 1904.

---

## NOTES.

THE seventeenth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Chicago on Saturday, April 22. Titles and abstracts of papers to be presented at this meeting should be in the hands of the Secretary of the Section, Professor THOMAS F. HOLGATE, 617 Hamline Street, Evanston, Ill., not later than April 8.

THE Macmillan Company is about to issue for the AMERICAN MATHEMATICAL SOCIETY a volume of 187 pages, of BULLETIN size, containing the lectures delivered before the

† Thomson, *loc. cit.*

Boston Colloquium, September 2–5, 1903, by H. S. WHITE: "Linear systems of curves on algebraic surfaces"; F. S. WOODS: "Forms of non-euclidean space"; E. B. VAN VLECK: "Selected topics in the theory of divergent series and of continued fractions." The price of the volume is $2.00; to members of the Society, $1.50. Subscriptions should be sent to The Macmillan Company, 66 Fifth Avenue, New York, N. Y.

THE annual meeting of the British mathematical association was held at King's College, Cambridge, January 28. Professor G. B. MATHEWS was elected president for the ensuing year. The following papers were read: by Mr. E. N. LANGLEY, "Models and their uses"; by Mr. W. H. WAGSTAFF, "On the new geometry"; by Professor A. R. FORSYTH and Mr. A. W. SIDDONS, "Should Greek be compulsory for candidates in mathematics at Cambridge?"

AT the meeting of the London mathematical society held on February 9 the following papers were read: by Dr. E. W. HOBSON, "General theory of transfinite numbers and order types"; by Mr. P. W. WOOD, "On the reducibility of covariants of binary quantics of infinite order, II."

A MEETING of the New York section of the Association of teachers of mathematics in the Middle States and Maryland was held March 4 at the College of the City of New York. A paper on "The graph in early algebra" was read by Professor J. Bowden, and Mr. Denbrigh, Mr. O. W. Anthony and Miss N. S. Ingalls, discussed "The first year of algebra." President J. H. Finley, of the College of the City of New York, addressed the meeting.

THE seventy-fifth annual meeting of the British association for the advancement of science will be held at Cape Town, South Africa, August 15–18, and at Johannesburg, August 29–31, under the presidency of Professor G. H. DARWIN. Applications for membership in the association must be received by May 31. The president of section A (mathematics and physics) is Professor A. R. FORSYTH.

THE Adams prize for 1904 was not awarded. The subject for the prize for 1906 (to be awarded in March 1907) is to dis-

cuss the inequalities of the moon's motion due to the direct action of the planets. Any one may compete for the prize who has at any time been admitted to a degree in Cambridge University. The essays must be sent to the vice-chancellor on or before December 16, 1906. The value of the prize is about 225 pounds sterling.

THE Smith's prizes for the present year have been awarded to H. BATEMAN, for his essay "The solution of linear differential equations by means of definite integrals," and to P. E. MARRACK, for his essay "Absorption by matter of Röntgen and $\gamma$-rays."

THE annual list of prize questions of the Dutch royal academy recently issued includes the following problem in mathematics: "In the case of constant curvature, the determination of the volume of the tetrahedron in elliptic space of three dimensions reduces to that of the hyperspace tetrahedron (extension of the notion of spherical trigonometry) in space of four dimensions. It is required to collect the literature relative to the determination of the latter volume and to extend the theory in some important point." (See the memoir of Schläfli, *Nieuw Archief voor Wiskunde*, 2d series, volume 6, 2d part, page 199.) The prize is 150 florins.

THE following books are announced as in the press of B. G. Teubner, Leipzig; Encyclopädie V, 1, 2, VI, 1, 1, VI, 2, 1; Encyclopédie I 1, 2, I, 2, 1, I, 3, 1, I, 4, 1; L. HEFFTER and C. KÖHLER, "Einführung in die analytische Geometrie der Ebene und des Raumes I"; W. F. OSGOOD, "Allgemeine Funktionentheorie, I"; E. SCHRÖDER, "Vorlesungen über die Algebra der Logik, II, 2"; O. STOLZ and J. GMEINER, "Funktionentheorie, II"; E. VIVANTI and A. GUTZMER, "Theorie der eindeutigen analytischen Funktionen"; J. THOMAE, "Formeln und Sätze aus dem Gebiete der elliptischen Funktionen, nebst Anwendungen."

THE following courses will be offered by the various universities during the summer semester, 1905. Lectures will begin about May 3, on account of the lateness of Easter.

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: Integral calculus, four hours; Applications of elliptic functions,

two hours; Hypergeometric series, two hours; Colloquium, two hours. — By Professor G. FROBENIUS: Theory of algebraic equations, II, four hours. — By Professor F. SCHOTTKY: Linear differential equations, four hours. — By Professor J. KNOBLAUCH: Theory and application of determinants, four hours; Surfaces and twisted curves, II, four hours; Selected chapters of elliptic functions, one hour. — By Professor R. LEHMANN-FILHÉS: Differential calculus, four hours; with exercises, one hour. — By Professor G. HETTNER: Transcendental nature of $\pi$ and $e$. — By DR. E. LANDAU: Introduction to the theory of functions, two hours. By Dr. I. SCHUR: Analytic geometry, four hours; Theory of numbers, II: algebraic and ideal numbers, four hours.

UNIVERSITY OF BERN. — By Professor J. H. GRAF: Bessel functions, three hours; Elliptic functions, three hours; Definite integrals and gamma functions, three hours; Theory of functions, two hours; Differential equations, II, two hours; Differential and integral calculus, two hours; Insurance and investments, two hours; Seminar, two hours. — By Professor E. OTT: Differential calculus, two hours; Analytic geometry of the plane, I, two hours; Analytic geometry of space, two hours. — By Professor G. SIDLER; Geometric exercises, two hours. — By Professor G. HUBER: Analytic geometry of quadrics, two hours. — By Dr. A. BENTELI: Descriptive geometry, four hours; with exercises, three hours. — By Dr. C. MOSER: Introduction to life-insurance, two hours; Seminar (with Professor Graf), two hours. — By Dr. L. CRELIER: Central projection, two hours; Synthetic geometry, two hours.

UNIVERSITY OF BONN. — By Professsr E. STUDY: Mechanics, II, three hours; Methods of geometric depiction, three hours. By Professor G. KOWALEWSKI: Theory of determinants, two hours; Theory of differential equations, four hours; with exercises, one hour; Theory of sets of points and foundations of the theory of functions, one hour. — By Professor FR. LONDON: Elements of the differential and integral calculus, four hours; with exercises, one hour; Descriptive geometry, II, with exercises in drawing, three hours. The Seminar will be conducted by Professors Study, Kowalewski and London, one hour.

UNIVERSITY OF BRESLAU. — By Professor J. ROSANES: Analytic geometry of the plane, four hours; Theory of in-

variants, two hours; Seminar, one hour. — By Professor R. STURM: Theory of geometric relations, four hours; Curves and surfaces of the third order, two hours; Seminar, two hours. — By Professor G. LANDSBERG: Integral calculus, four hours; with exercises, one hour. — By Professor A. KNESER: Encyclopedia of elementary mathematics, four hours; Partial differential equations of mathematical physics, four hours; Seminar, two hours; Seminar for beginners, two hours.

UNIVERSITY OF FREIBURG. — By Professor J. LÜROTH: Elliptic functions, four and one half hours; Numerical calculation, two hours. — By Professor L. STICKELBERGER: Integral calculus with exercises, five hours; Infinitesimal geometry, three hours; Seminar, two hours. — By Professor A. LOEWY: Algebraic equations, four hours; Foundations of geometry, two hours. — By Dr. K. SEITH: Elementary treatment of conics, two hours.

UNIVERSITY OF HALLE. — By Professor G. CANTOR: Analytic mechanics, five hours; Descriptive geometry, four hours; Seminar, two hours. — By Professor A. WANGERIN: Elliptic functions, five hours; Spherical astronomy and mathematical geography, three hours: Conformal depiction, one hour; Seminar, two hours. — By Professor V. EBERHARD: Differential calculus, four hours; with exercises, one hour. — By Professor F. BUCHHOLZ: Applications of the calculus of variations to higher mechanics, four hours. — By Dr. F. BERNSTEIN: Analytic geometry of the plane, three hours; Elements of the theory of differential equations, two hours.

UNIVERSITY OF HEIDELBERG. — By Professor L. KOENIGSBERGER: Differential and integral calculus, four hours; Theory of functions, four hours; Seminar, two hours. — By Professor E. VALENTINER: Theory of planetary orbits, four hours. — By Professor M. CANTOR: Algebraic analysis, four hours; Arithmetic and algebra for students in political economy, three hours. — By Professor K. KOEHLER: Analytic geometry of the plane, three hours; Descriptive geometry, two hours. — By Professor K. BOEHM: Elementary mathematics (geometry), four hours; Helmholtz's theory of vortex motion, two hours.

UNIVERSITY OF JENA. — By Professor G. FREGE: Mechanics, I, four hours; Seminar, two hours. — By Professor A.

GUTZMER: Differential calculus with exercises, four hours; Differential equations, four hours. — By Professor J. THOMAE: Plane analytic geometry, four hours; Ultra-elliptic functions, two hours. — By Professor R. RAU: Technical mechanics, four hours; Seminar in drawing, three hours.

UNIVERSITY OF KIEL. — By Professor L. POCHHAMMER: Analytic geometry of the plane, four hours; Theory of definite integrals, four hours; Seminar, one hour. Professor Stäckel's successor has not been announced, but the following courses will be given: Differential calculus and introduction to higher analysis, four hours; Analytic mechanics, four hours; Seminar, one and one half hours. — By Dr. E. WEINNOLDT: Descriptive geometry, four hours.

UNIVERSITY OF LEIPZIG. — By Professor C. NEUMANN: Selected chapters of mathematical physics, four hours; Seminar, one hour. — By Professor A. MAYER: Ordinary differential equations, four hours; with exercises, one hour. — By Professor O. HÖLDER: Introduction to the theory of algebraic equations, two hours; General theory of functions, four hours; Seminar, one hour. — By Professor K. ROHN: Analytic geometry of the plane, four hours; with exercises, one hour; Descriptive geometry, I, two hours; with exercises, two hours. — By Professor E. VON OETTINGEN: Geometric perspective drawing, two hours. — By Professor F. HAUSDORFF: Differential geometry, four hours. — By Professor H. LIEBMANN: Theory of definite integrals, three hours; Noneuclidean geometry, two hours. — By Professor O. FISCHER: Introduction to the mathematical treatment of the natural sciences, three hours. — By Professor H. BRUNS: Theory of probabilities, four hours; Seminar for numerical calculation, two hours.

UNIVERSITY OF MUNICH. — By Professor F. LINDEMANN: Analytic geometry of space, five hours; Definite integrals and Fourier's series, four hours; Seminar, two hours. — By Professor A. VOSS: Introduction to the theory of invariants, four hours; Selected chapters in the theory of surfaces, four hours; Seminar, two hours. — By Professor A. PRINGSHEIM: Integral calculus, five hours; exercises and applications, two hours. — By Professor L. GRAETZ; Analytic mechanics, II., four hours; Theory of heat and kinetic gas theory, four hours. — By Professor K. DOEHLEMANN: Descriptive geometry, II., three

hours; with exercises, two hours; Graphical statics, with exercises, three hours. — By Professor E. v. WEBER: Differential calculus, four hours; with exercises, two hours; Algebraic analysis, four hours. — By Dr. H. BRUNN: Elements of higher mathematics for students of all faculties, four hours.

UNIVERSITY OF WÜRZBURG. — By Professor F. PRYM: Theory of ordinary differential equations, four hours; Proseminar on the calculus, two hours; Seminar on theory of functions, two hours. — By Professor E. SELLING: Differential and integral calculus, four hours; Probabilities and adjustment of errors, two hours; Insurance, two hours. — By Professor G. ROST: Analytic and synthetic geometry of conics, four hours; Descriptive geometry, II, four hours; Analytic mechanics, II, four hours; Algebraic curves, two hours; Seminar, four hours.

DURING the year 1904 the following doctorates were conferred in mathematics by the university of Paris (the title of the dissertation is given in each case): R. D'ADHÉMAR, "On a class of partial differential equations of the second order, of hyperbolic type, in three or four independent variables"; S. BERNSTEIN, "On the analytic nature of the solutions of partial differential equations of the second order"; E. ESCLANGON, "Quasi-periodic functions"; S. POTRON, "The groups of order $p^6$"; P. VANDEUREN, "Theory of continuous bilinear fields"; G. DUMAS, "On the functions of algebraic character in the region of a given point."

ON February 13 occurred the centenary of the birth of G. LEJEUNE DIRICHLET. The day was appropriately celebrated by the mathematical society of Göttingen, before which Professor H. Minkowski gave an address on the position of Dirichlet in mathematics. The current number (number 1, volume 129) of Crelle's *Journal* is also dedicated to the memory of Dirichlet. It contains a portrait drawing of Dirichlet on his deathbed, besides memoirs on the theory of numbers by Dedekind, Hilbert, Hensel and Mirimanoff.

PROFESSOR P. STÄCKEL, of the University of Kiel, has been appointed professor of mathematics at the technical school at Hanover.

PROFESSOR ROTH, professor emeritus of the University of Strassburg, has been decorated with the order of the crown of the third class.

DR. E. JAHNKE has been appointed professor of mathematics

in the mining academy at Berlin, as successor to Professor A. Kneser.

DR. A. MACFARLANE is preparing an extensive history of British mathematicians of the nineteenth century for the *Abhandlungen zur Geschichte der Mathematik*. He will deliver a course of six lectures on the same subject at Lehigh University, April 7 to April 17.

AT the Ecole Polytechnique, of Paris, Dr. M. PAINLEVÉ has been appointed professor of mechanics, as successor to Professor Léauté, who has retired with the title of honorary professor.

PROFESSOR P. BARBARIN, of Bordeaux, is preparing a French translation of Professor G. B. Halsted's Rational Geometry.

DR. E. BOREL has been appointed professor of the theory of functions at the University of Paris.

PROFESSOR M. LERCH, of the University of Freiburg, Switzerland, has been elected a corresponding member of the royal society of Liège.

THE honorary degree of doctor of science has been conferred upon Professor R. S. WOODWARD, of Columbia University, by the University of Pennsylvania.

AT Princeton University, Professor E. O. LOVETT has been appointed professor of astronomy, as successor to Professor C. A. YOUNG, who has been made professor emeritus.

PROFESSOR W. C. ESTY, Walker professor of mathematics at Amherst College, has resigned, after forty three-years of service. He will be succeeded by his son, Professor T. C. ESTY, of the University of Rochester.

MR. ROBERT TUCKER, editor of Clifford's mathematical papers, and secretary of the London mathematical society for thirty-five years, died January 29, at the age of 72 years.

PROFESSOR GUIDO HAUCK, of the technical school of Charlottenburg, well known for his writings on perspective and constructive geometry, died January 25 at the age of 59 years.

THE death is also announced of Professor J. C. V. HOFFMANN, founder of the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, on January 21, at the age of 80 years.

# NEW PUBLICATIONS.

## I. HIGHER MATHEMATICS.

BASS (E. W.). Elements of differential calculus. 3rd edition, revised. New York, Wiley, 1904. 12mo. 8 + 356 pp. Cloth. $4.00

BLANCKMEISTER (M.). Ueber die Hamilton'sche Funktion als Grundlage der geometrischen Abbildungstheorie, insbesondere über die Herleitung der allgemeinsten Bedingung der aberrationsfreien Abbildung von Punktpaaren. Rostock, 1904. 8vo. 53 pp. M. 2.00

BROCKMANN (F.). Zur Theorie der Linienflächen 2. Ordnung. Rostock, 1904. 8vo. 74 pp. M. 2.00

BRUNN (H.). Beziehungen des Du Bois-Reymondschen Mittelwertsatzes zur Ovaltheorie. Eine mathematische Studie. Berlin, Reimer, 1905. 4to. 10 + 138 pp. M. 7.00

GALE (A. S.). See SMITH (P. F.).

GEISSLER (K.). Die Kegelschnitte und ihr Zusammenhang durch die Continuität der Weitenbehaftungen. Mit einer Einführung in die Lehre von den Weitenbehaftungen. Jena, 1905. 8vo. M. 5.00

HASKELL (M. W.). The construction of conics under given conditions. (*Bulletin of the American Mathematical Society*, Vol. XI, pp. 268–273). New York, 1905.

KEYSER (C. J.). The universe and beyond; the existence of the hypercosmic. (*The Hibbert Journal*, Vol. 3, 1905, pp. 300–314.)

——. The axiom of infinity. (*The Hibbert Journal*, Vol. 3, 1905, pp. 380–383.)

MANDART (H.). Cours de géométrie analytique à deux dimensions (sections coniques). Namur, 1904. 8vo. 8 + 574 pp. F. 10.00

SMITH (P. F.) and GALE (A.S.). Introduction to analytic geometry. A condensation of "Elements of analytic geometry." Boston, Ginn, 1905. 8vo. Cloth. $1.25

TANNERY (J.). Introduction à la théorie des fonctions d'une variable. 2e édition, entièrement refondue. (En 2 volumes.) Vol. I. Paris, 1904. 8vo. 4 + 430 pp. Fr. 15.00

## II. ELEMENTARY MATHEMATICS.

AMALDI (U.). See ENRIQUES (F.).

BAKER (W. M.) and BOURNE (A. A.). Elementary algebra, Part 2. Teachers' edition. London, Bell, 1905. 8vo. Cloth. 5s.

BALTIN (R.) und MAIWALD (W.). Sammlung von Aufgaben aus der Arithmetik, Trigonometrie und Stereometrie, mit zahlreichen Anwendungen aus der Planimetrie und Physik für Seminare und Präparandenanstalten. Unter Zugrundelegung der Müller-Kutnewskyschen Aufgabensammlung, Teil 1, nach den preussischen Lehrplänen von 1901 bearbeitet. 1. Teil, Für die 1. Klasse der

Präparandenanstalten. 2te verbesserte Auflage. Leipzig, Teubner, 1904. 8vo. 6 + 110 pp. M. 1.40

BEHRENDSEN (O.). See BEITRÄGE.

BEITRÄGE, neue, zur Frage des mathematischen und physikalischen Unterrichts an den höheren Schulen. Vorträge, gehalten bei Gelegenheit des Ferienkurses für Oberlehrer der Mathematik und Physik, Göttingen, Ostern, 1904. Gesammelt und herausgegeben von F. Klein und E. Riecke. Mit einen Abdruck verschiedener einschlägiger Aufsätze von E. Götting und F. Klein. 1. Teil: Beiträge von O. Behrendsen, E. Bose, E. Götting, F. Klein, E. Riecke, J. Stark, K. Schwarzschild. Leipzig, Teubner, 1904. 8vo. 7 + 190 pp. M. 3.60

BOSE (E.). See BEITRÄGE.

BOURNE (A. A.). See BAKER (W. M.).

CRAIG (E. S.). See OXFORD.

DALE (J. B.). Logarithmic and trigonometric tables (five places). London, Arnold, 1905. 8vo. Cloth. 2s.

ENRIQUES (F.) e AMALDI (U.). Elementi di geometria ad uso delle scuole secondarie superiori. 2a edizione riveduta e semplificata. Bologna, Zanichelli, 1905. 16mo. 10 + 567 pp. Fr. 4.50

ERDMANN (K.), Anfangsgründe der ebenen Geometrie verbunden mit einer Aufgabensammlung. 2. (Schluss-) Teil. Dresden, Bleyl, 1905. 8vo. 7 + 164 pp. M. 2.25

GÖRTH (H.). See MATTIAT (D.).

GÖTTING (E.). See BEITRÄGE.

HALL (H. S.) and STEVENS (F. H.). Lessons in experimental and practical geometry. London, Macmillan, 1905. 8vo. Cloth. 1s. 6d.

HENSELIN (A.). Rechentafel. Das grosse Einmaleins bis 999 mal 999 nebst einer Kreisberechnungstabelle. 2te Auflage. Berlin, 1904. 4to. Cloth. M. 6.00

HOPKINS (G. I.). Teachers' manual to "Inductive plane geometry." Boston, Heath, 1905. 16mo. 107 pp. $0.40

KLEIN (F.). See BEITRÄGE.

MAIWALD (W.). See BALTIN (R.).

MARCOU (C. A.). See OXFORD.

MATEK (B.). Resultate der Aufgaben in Močnik-Spielmanns Lehrbuch der Geometrie für die oberen Klassen der Mittelschulen. Ausgegeben für Gymnasien. Wien, Gerold, 1905. 8vo. 4 + 128 pp. M. 2.40

MATTIAT (D.). Die Raumlehre in der Volks- und Fortbildungsschule. Als Leitfaden und Wiederholungsbuch herausgegeben. Bevorwortet von A. Görth. 5te, erweiterte und verbesserte Auflage. Nach neuester Orthographie. Leipzig, Hofmann, 1904. 8vo. 8 + 90 pp. M. 0.80

MAYER (J. E.). Das mathematische Pensum des Primaners. Ein

Hilfsbuch für den Primaner humanistischer und realistischer Gymnasien, sowie für das Selbststudium. 14. und 15. Heft. Leipzig, Schäfer, 1905. 8vo. M. 1.00

NANNEI (E.). Elementi di geometria. 2a edizione. Parte I: Planimetria. Puntata 1. Milano, Vallardi, 1905. 16mo. Pp. 1–128. (Collezione di manuali scientifici, storici e letterari.) Fr. 3.00

OXFORD questions in algebra. Papers set in pass moderations. Answers by C. A. Marcou and E. S. Craig. 3rd series, from Michaelmas term, 1899, to Michaelmas term, 1904. London, Simpkin, 1905. 12mo. Cloth. 2s. 6d.

PINCHERLE (S.). Algebra elementare. 9a edizione riveduta. Milano, Hoepli, 1905. 16mo. 8 + 210 pp. (Manuali Hoepli.)

REINHARD (P.). Rechnungstabelle. Tableau de calcul. 3te Auflage. Bern, 1904. Folio, 2 pages, on cloth. M. 5.00

RIBI (D.). Aufgaben über die Elemente der Algebra, methodisch geordnet und in engem Anschluss an den "Leitfaden" von M. Zwicky. III. Heft. 8te Auflage. Herausgegeben von G. Wernly. Bern, Francke, 1904. 8vo. 32 pp. M. 0.50

RIECKE (E.), SCHILLING (F.), SCHWARZSCHILD (K.), STARK (J.). See BEITRÄGE.

STEVENS (F. H.). See HALL (H. S.).

SUÁREZ SOMONTE (J.). Algebra y trigonometría. 2a edicion. Madrid, 1904. 8vo. 6 + 246 pp. M. 10.00

TANNER (J. H.). Key to "Elementary algebra." New York, American Book Co., 1905. 12mo. 128 + 23 pp. Half-leather. $1.00

WERNLY (G.), ZWICKY (M.). See RIBI (D.).

## III. APPLIED MATHEMATICS.

ARIÈS (E.). La statique chimique basée sur les deux principes fondamentaux de la thermodynamique. Paris, 1904. 8vo. 8 + 250 pp. F. 10.00

BERRY (C. W.). The temperature-entropy diagram. New York, Wiley, 1905. 12mo. 16 + 134 pp. Cloth. $1.25

CHARLIER (C. W.). Die Mechanik des Himmels. Vorlesungen. 2. Band, 1. Abteilung. Leipzig, Veit, 1905. 8vo. 320 pp. M. 12.00

CLARK (J. B.). Mathematical and physical tables. London, Oliver, 1905. 8vo. 6d.

CUNYNGHAME (H.). Geometrical political economy. An elementary treatise on the method of explaining some of the theories of pure economic science by means of diagrams. Oxford, 1904. 8vo. Cloth. 2s. 6d.

DUAIME (H.). See SCHILLING (F.).

FARROW (F. R.). Stresses and strains: their calculation and that of their resistances, by formulæ and graphic methods. New York, Macmillan, 1905. (*The Builder*, students' series.) 12mo. 6 + 144 pp. Cloth. $1.60

FISH (J. C. L.). Descriptive geometry. New York, Macmillan, 1904. 12mo. 64 pp. Cloth. $0.90

FOERSTER (W.). Astrometrie oder die Lehre von der Ortsbestimmung im Himmelsraume, zugleich als Grundlage aller Zeit- und Raummessung. 1. Heft: Die Sphärik und die Koordinatensysteme, sowie die Bezeichnungen und die sphärischen Koordinatenmessungen. Berlin, Reimer, 1905. 8vo. 160 pp. M. 4.00

HAMMER (E.). Der logarithmische Rechenschieber und sein Gebrauch. Eine elementare Anleitung zur Verwendung des Instruments für Studierende und für Praktiker. 3te, durchgesehene Auflage. Stuttgart, Metzler, 1904. 8vo. 8 + 71 pp. M. 0.70

HECK (R. C. H.). The steam-engine and other steam-motors: a textbook for engineering colleges and a treatise for engineers. (In 2 vols.) Vol. 1: The thermodynamics and the mechanics of the engine. New York, Van Nostrand, 1905. 8vo. 8 + 391 pp. Cloth. $3.50

HEYGENDORFF (W.). Ueber das Verteilungs-Gleichgewicht der Ionen. (Diss.). Leipzig, Schlemminger, 1904. 8vo. 32 pp. M. 0.75

HÖFLER (A.). Physik mit Zusätzen aus der angewandten Mathematik, aus der Logik und Psychologie und mit 230 physikalischen Leitaufgaben. Verfasst unter Mitwirkung von E. Maiss und F. Poske. Braunschweig, 1904. 8vo. 31 + 966 pp. M. 15.00

JAUMANN (G.). Die Grundlagen der Bewegungslehre, von einem modernen Standpunkte aus dargestellt. Leipzig, Barth, 1905. 8vo. 6 + 421 pp. Cloth. M. 12.00

MAISS (E.). See HÖFLER (A.).

MAURER (E. R.). Technical mechanics: a textbook for technical colleges. 2d edition, revised and enlarged. New York, Wiley, 1904. 12mo. 17 + 409 pp. Cloth. $4.00

MERRIMAN (M.). Elements of precise surveying and geodesy. 2d edition. New York, Wiley, 1904. 8vo. 261 pp. Cloth. $2.50

——. Elements of mechanics. Forty lessons for beginners in engineering. New York, Wiley, 1905. 12mo. 172 pp. Cloth. $1.00

POSKE (F.). See HÖFLER (A.).

SCHILLING (F.). Nouveaux modèles cinématiques et introduction nouvelle à la théorie des courbes cycloïdales. Traduit par H. Duaime. Halle, 1904. 18 pp. M. 0.60